Multi-Scale Image Reconstruction with Low-Count Poisson Data

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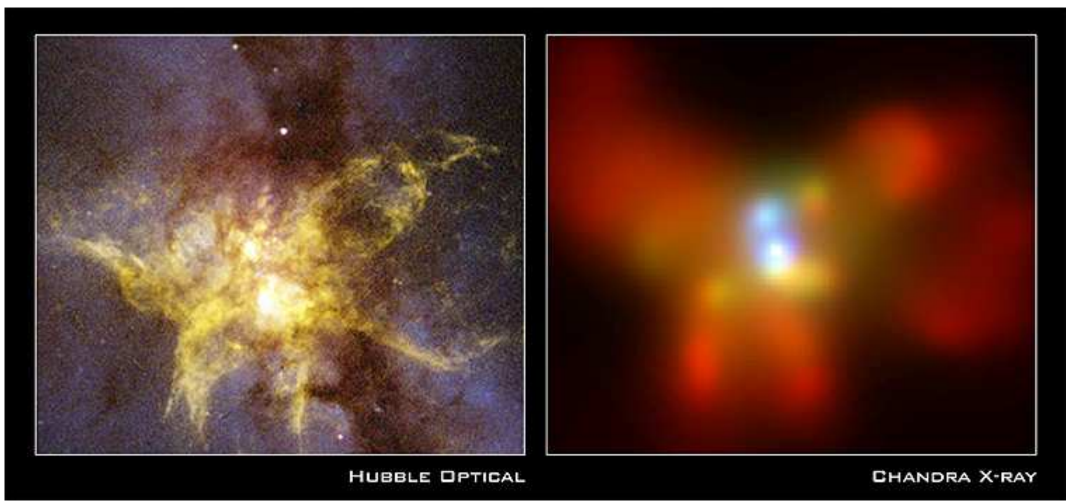


[a]Joint Work with the California-Harvard Astrostatistical Collaboration

Automate, Formulate, and Evaluate

Three Goals for Low-Count Image Analysis

Automate: We would like to automate

1. *model fitting* to avoid subjective stopping rules used to control reconstruction quality, and
2. *the search for structure* to avoid choosing smoothing parameters to enhance supposed structure in the reconstruction.

Formulate: We would like to formulate low-count image analysis in the terms of well understood *statistical theory* in order to better understand the characteristics and properties of image analysis methods and results.

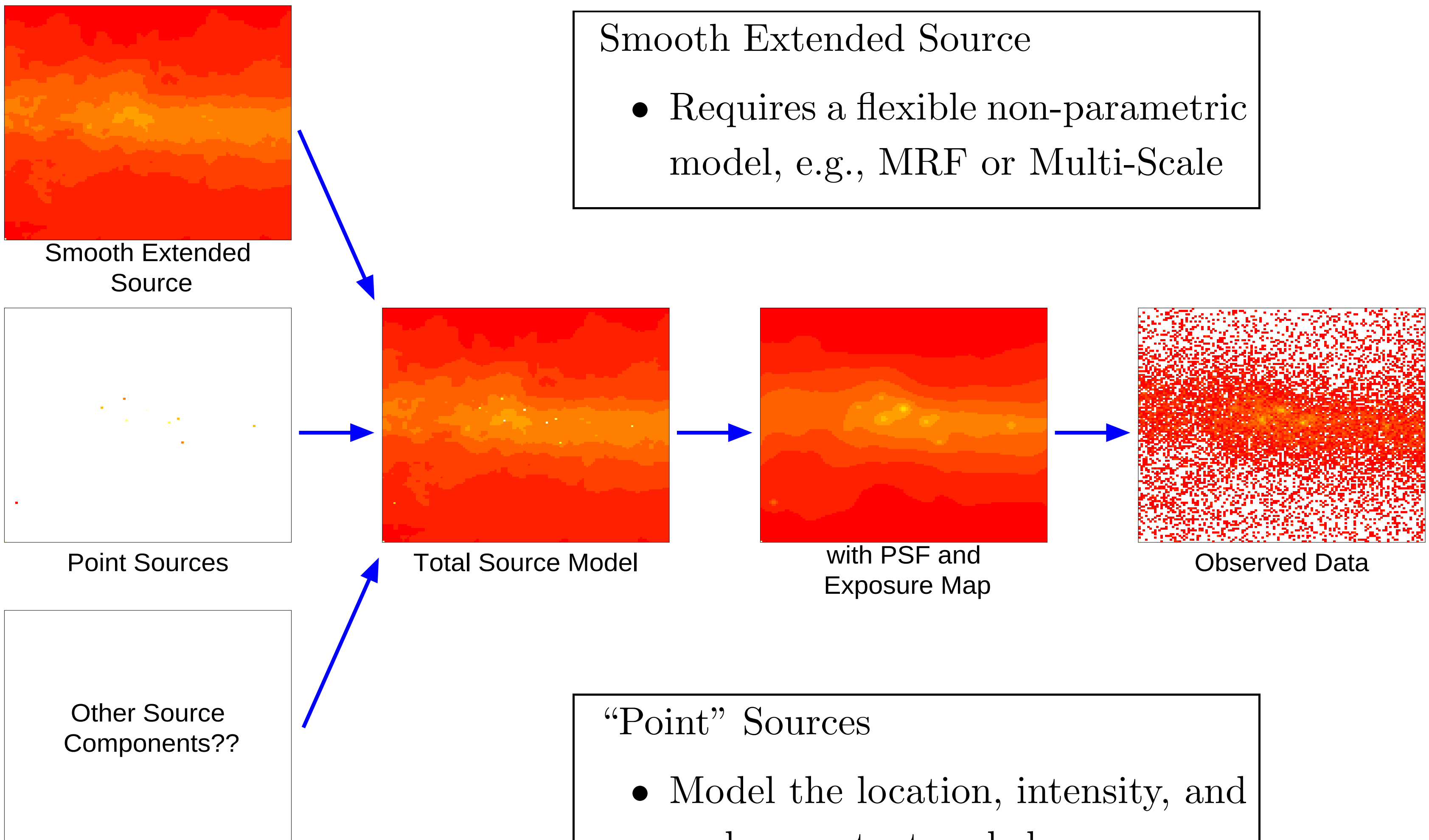
Evaluate: We would like to evaluate

1. the *statistical error* in the fitted reconstruction under the assumed model,
2. the likelihood that supposed or proposed structures exist in the astronomical source, and
3. *the plausibility of the model assumptions.*

A Statistical Model for the Data Generation Process

Smooth Extended Source

- Requires a flexible non-parametric model, e.g., MRF or Multi-Scale



“Point” Sources

- Model the location, intensity, and perhaps extent and shape.

Bayesian Deconvolution

Blurring Matrix
* known from calibration

Source Model

$$\boldsymbol{\lambda} = \mathbf{PA}\boldsymbol{\mu} + \boldsymbol{\xi},$$

Expected Photon Count

Non−Homogeneous Stochastic Censoring
* known from calibration

Background Contamination
* often fit using background observation

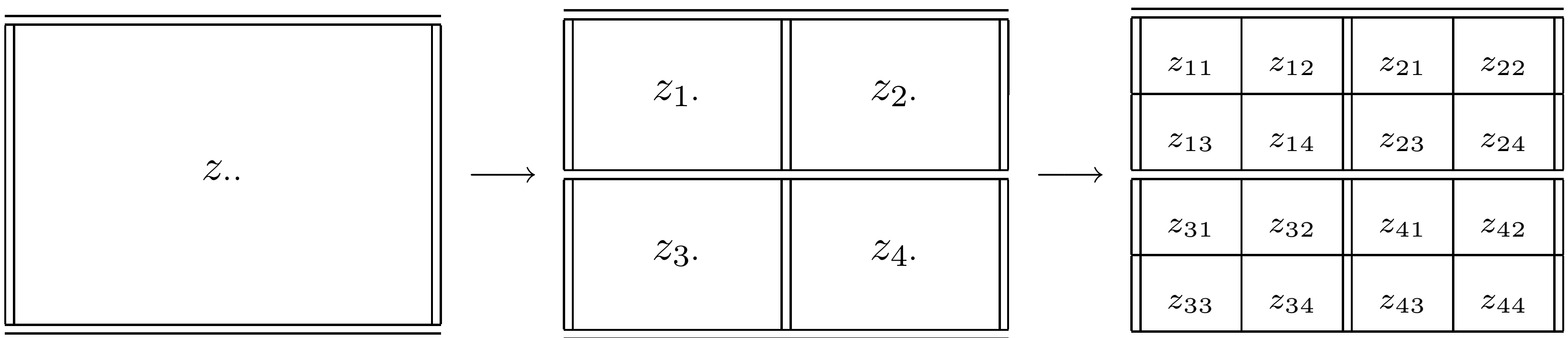
- The counts are modeled as indep. Poisson variables with means given by $\boldsymbol{\lambda}$.
- $\boldsymbol{\mu}$ may be the sum of several components.
- We focus on idealized counts from one of these components:

$$Z_i \sim \text{Poisson}(\mu_i)$$

A Smoothing Muliscale Prior for an Extended Source

<u>The Nowak-Kolaczyk Multiscale Model:</u>

Low Resolution *High Resolution*



$z_{\cdot\cdot} \sim \text{Poisson}(\mu)$ $\quad$ $\boldsymbol{z}_{,\cdot}|z_{\cdot\cdot} \sim \text{Multinomial}(\boldsymbol{p}_1)$ $\quad$ $\boldsymbol{z}_{i,}|z_{i\cdot} \sim \text{Multinomial}(\boldsymbol{p}_{2i})$

$\mu \sim \text{Gamma}\{(\alpha_0, \beta_1)\}$ $\quad$ $\boldsymbol{p}_1 \sim \text{Dirich.}\{(\alpha_1, \alpha_1, \alpha_1, \alpha_1)\}$ $\quad$ $\boldsymbol{p}_{2i} \sim \text{Dirich.}\{(\alpha_2, \alpha_2, \alpha_2, \alpha_2)\}$

- We use a multi-level model to fit the smoothing parameter $\boldsymbol{\alpha}$.
- Cycle-Spinning eliminates the effect of the choice of coordinates.

Wavelet like model in a fully Bayesian analysis.

A Smoothing Markov Random Field Prior Distribution

μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{16}
μ_{21}	μ_{22}	μ_{23}	μ_{24}	μ_{25}	μ_{26}
μ_{31}	$\boldsymbol{\mu}_{32}$	$\boldsymbol{\mu}_{33}$	$\boldsymbol{\mu}_{34}$	μ_{35}	μ_{36}
μ_{41}	$\boldsymbol{\mu}_{42}$	$\boxed{\boldsymbol{\mu}_{43}}$	$\boldsymbol{\mu}_{44}$	μ_{45}	μ_{46}
μ_{51}	$\boldsymbol{\mu}_{52}$	$\boldsymbol{\mu}_{53}$	$\boldsymbol{\mu}_{54}$	μ_{55}	μ_{56}
μ_{61}	μ_{62}	μ_{63}	μ_{64}	μ_{65}	μ_{66}

$$\log(\mu_{43}) \sim N\Big(\text{mean of log neighbors},\ \text{smoothing parameter}\Big)$$

Explicitly quantifies smoothness of the image.

The Advantages of A Model-Based Statistical Formulation

1. The use of well defined statistical estimates such as ML estimates, MAP estimates, or posterior means, eliminates the need for ad-hoc stopping rules (Esch et al., ApJ, 2004).
2. Statistical theory lends insight into the computation of statistical errors with Bayesian and/or frequency properties (Esch et al., ApJ, 2004).
3. Allow us incorporate knowledge from other data sources (Slides 8 - 12).
4. Principled methods for comparing and/or evaluating models (Alanna Connors's talk).
5. Quantify evidence for supposed structure under flexible model (Slides 13 & 14).

Incorporating Outside Information

Outside information can be critical with low-count data. Lucky, such information is often forthcoming in the form of high-count high-resolution data from a different energy band (e.g., Optical or Radio).

Incorporating Information Through Model Components

Setting Model Parameters

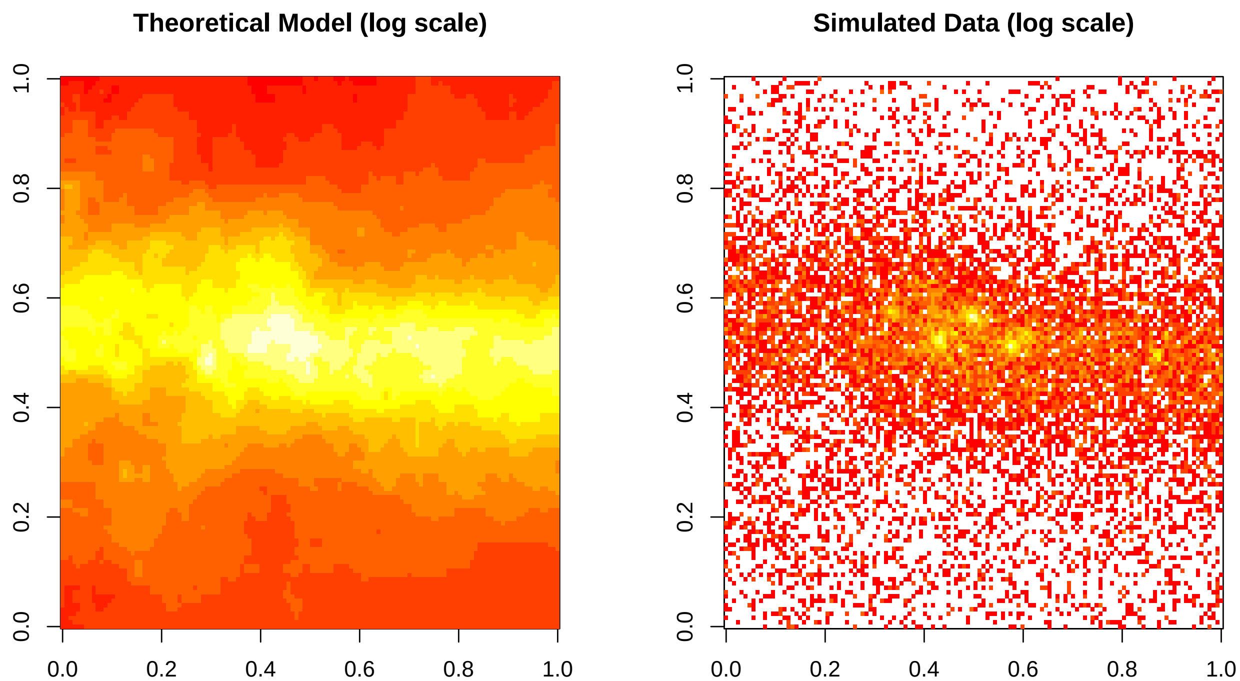
- The number of and location of point sources.
- Smoothing parameters for extended source. **I'll focus on this.**
- A characterization of how smoothing parameters vary across source.

Incorporating Information Through Bayesian Prior Distributions

A More Flexible Approach

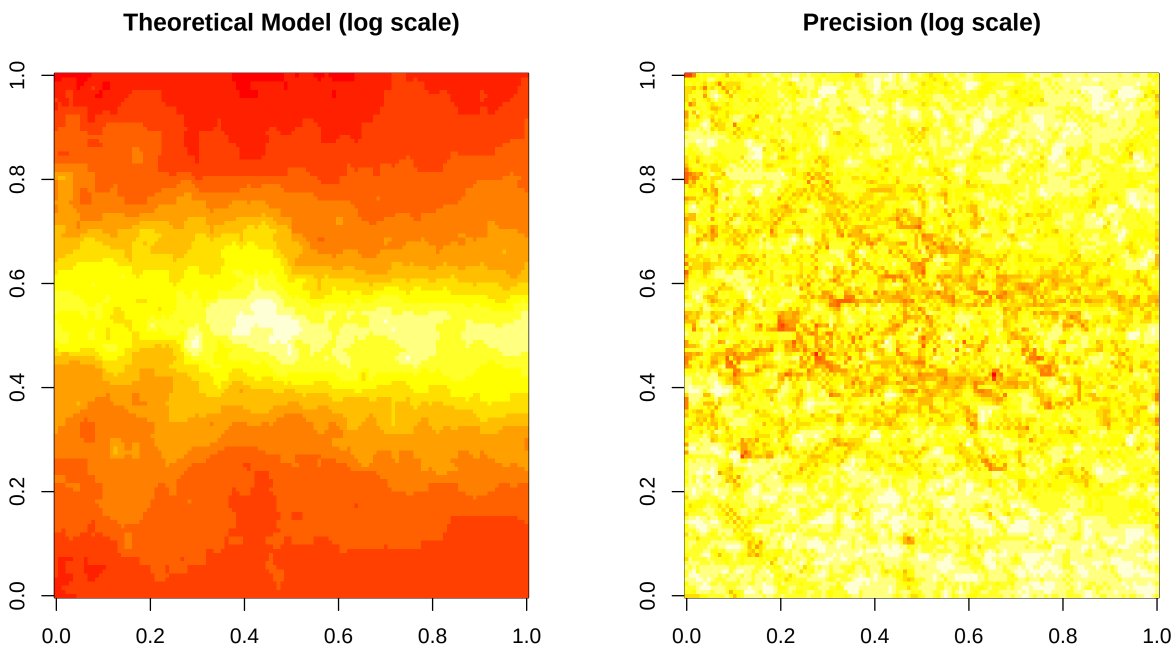
- Include a region where a point source is likely to exist.
- Encourage parameters to be *similar* to those obtained from better data.

Example: Cygnus Region Simulation



- The image on the left is based on a theoretical model of the Galactic Diffuse Emission along with several point sources.
- The image on the right is from a Glast simulator under this model.

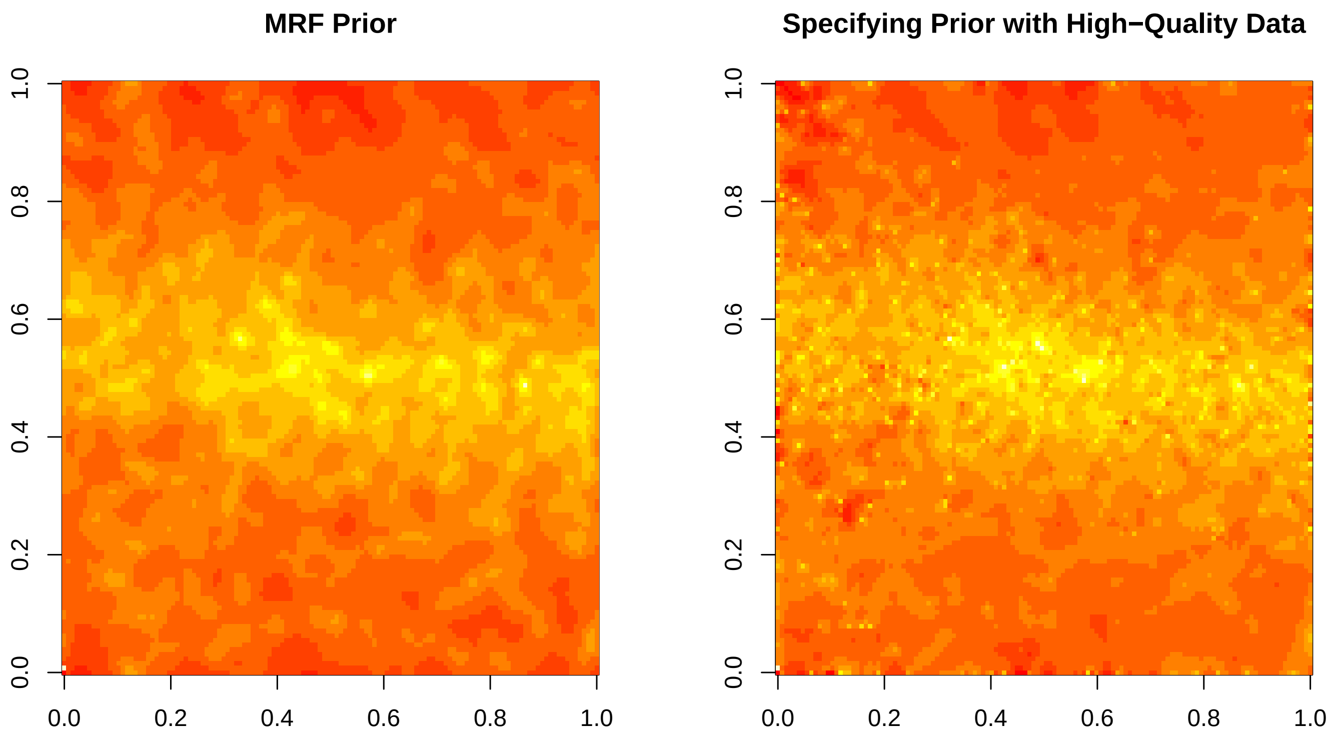
Varying the MRF Smoothing Parameter Across an Image



- Let Υ be the log of the high quality image and $\omega_{ij} = \kappa/(\Upsilon_i - \Upsilon_j)$.
- We use the MRF prior on $\upsilon = \log(\mu)$

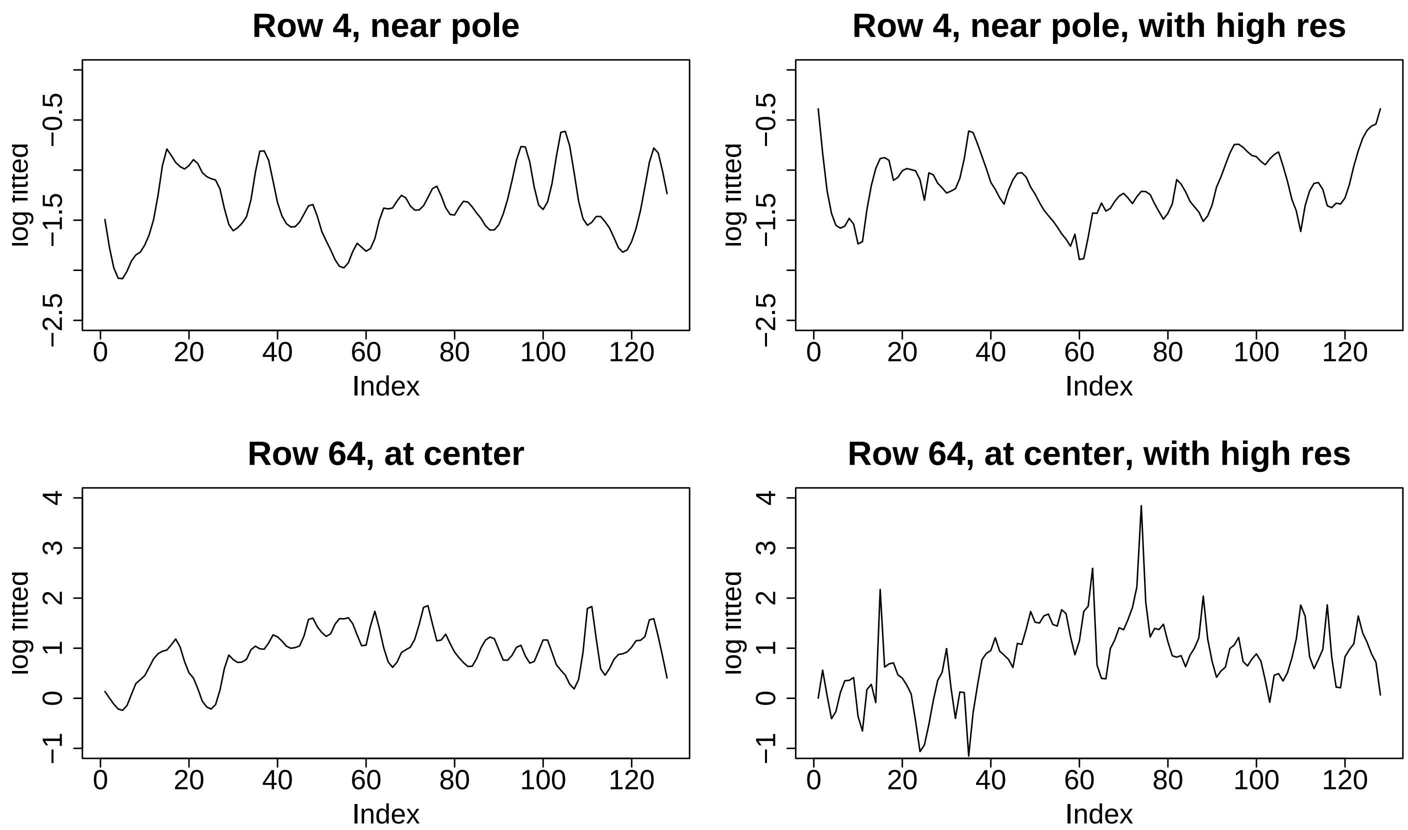
$$\upsilon_i \sim N\left(\frac{\sum_{j\in\mathcal{J}(i)} \omega_{ij}\upsilon_j}{\sum_{j\in\mathcal{J}(i)} \omega_{ij}},\ \frac{1}{\sum_{j\in\mathcal{J}(i)} \omega_{ij}}\right)$$

Results: Smoother at the Poles



- We under smooth to highlight effect.
- Similar smoothness near the poles.
- MRF alone is much smoother near the center.

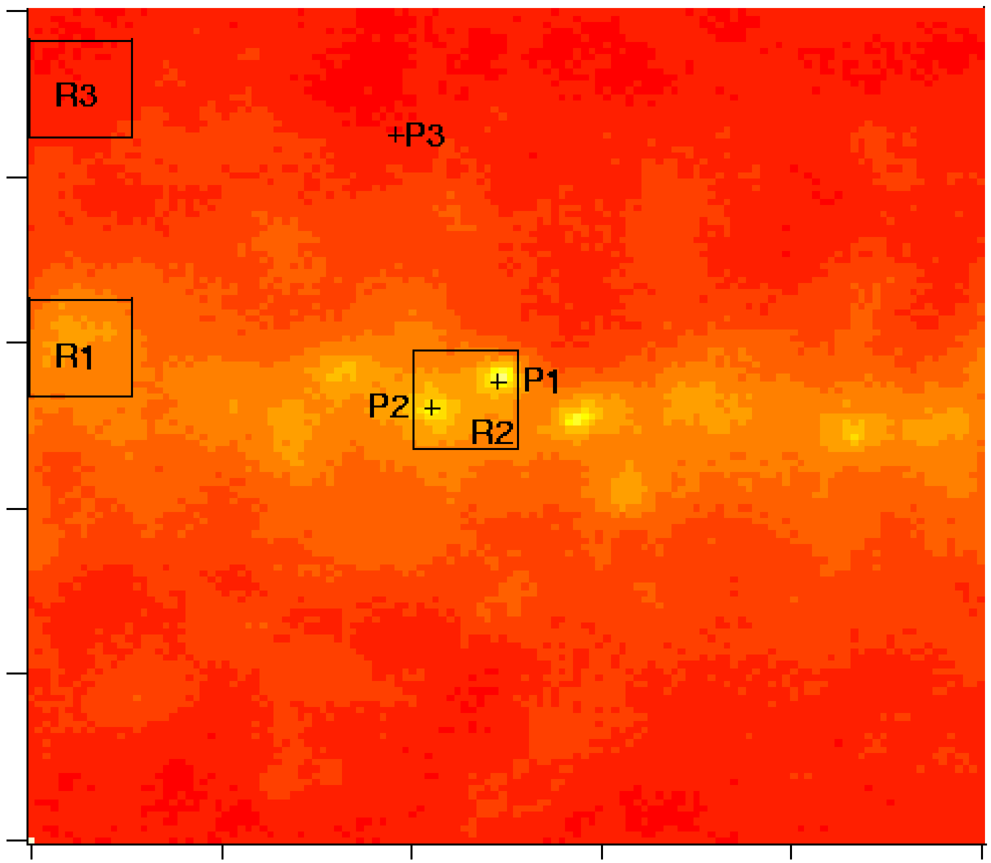
Results: A Closer Look



- Notice thie similar smoothness in the first two plots.
- But the MRF is much smoother in the second two plots.

Looking for Structure

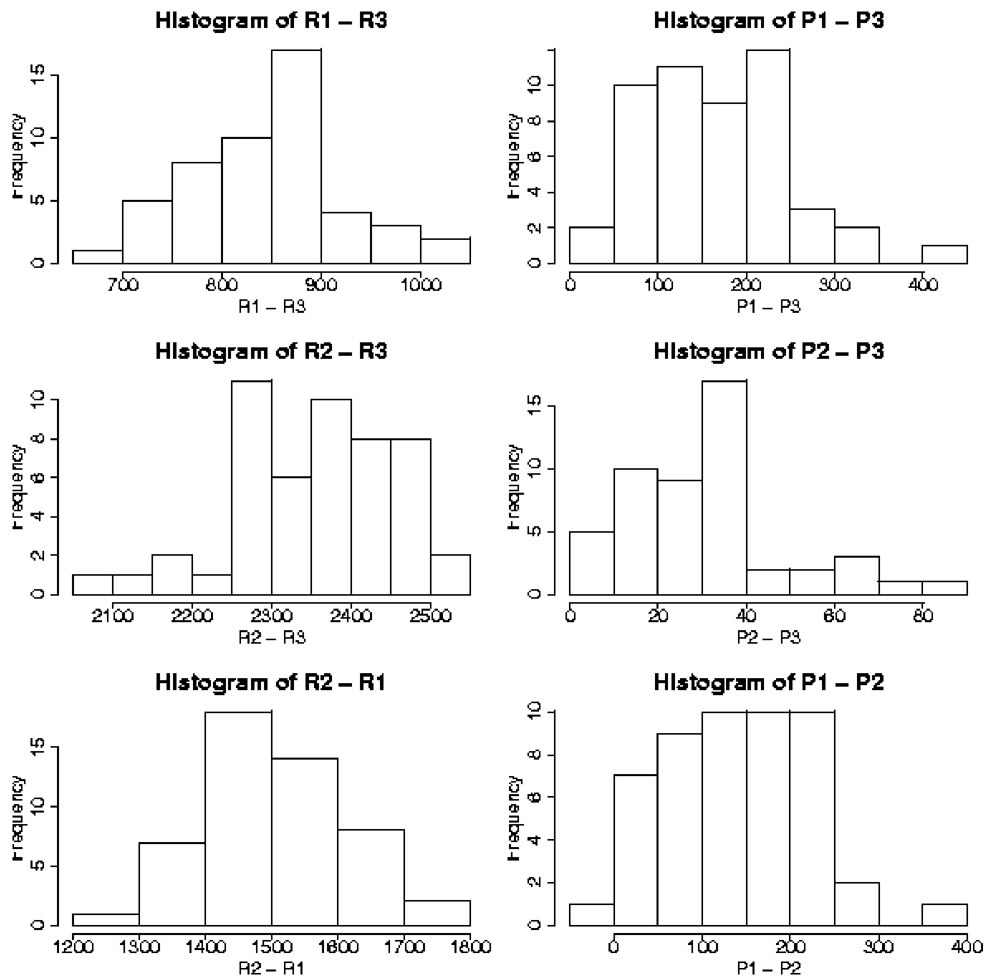
Once we have settled on a model, we can use statistical tools to investigate structure in the image.



Is the apparent local maximum in R1 real?

Is the P1 brighter than P2?

Some Posterior Probabilities



Monte Carlo Evaluation of Some Posterior Probabilities.

Challenges

Model Identifiability

- Point source vs. glob in extended source
- Requires outside information for smoothing parameters

Prior Specification

- Results can depend on choices
- Requires external evaluation of prior and/or results

Statistical Computation

- Subtle methods are required
- Expensive in CPU time

Complex scientific questions require sophisticated statistical solutions

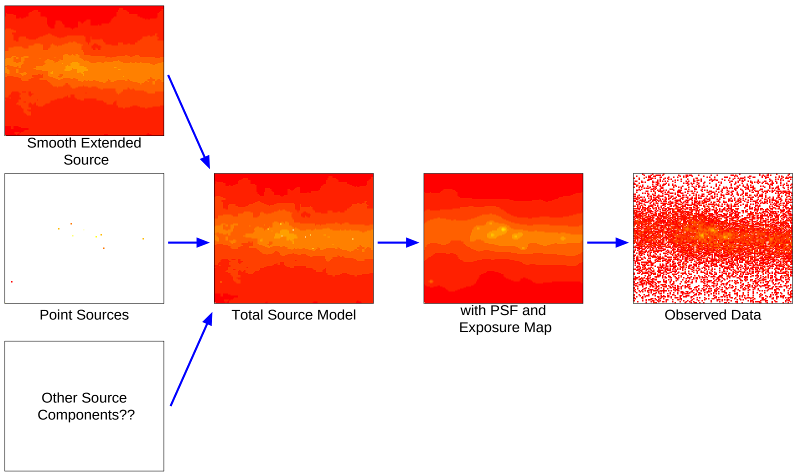
Model-based methods offer much promise,

But many challenges remain.

Comparing/Evaluating Models I

Alanna's Question: Does one model component suffice, or are two necessary?

The difficult task of fitting the number of components in a *finite mixture model.*



Some Statistical Strategies

Confidence Intervals: Fit the full model, and, compute confidence intervals, e.g., for the total expected count from second component.

Residuals: Fit the smaller model, and, compute residuals around the fitted reconstruction along with their prediction intervals. Is there evidence that the residuals are too big or vary in an unexpected systematic way?

Significance Tests: Likelihood ratio and other tests can be used to compare one and two component models. Although the standard χ^2 distribution is not appropriate, the test can be calibrated using Monte Carlo (e.g., Protassov et al., 2002).

Comparing/Evaluating Models II

Using a Bayesian prior distribution to formulate a frequentist significance test.

A procedure:

1. Construct a prior distribution that favors a null hypothesis

   H_0: *object is a point source*

2. Compute the posterior distribution and evaluate the propensity of the alternative hypothesis

   H_A: *object is an extended source*

3. Using a test statistic, prior parameters can be used to set the level (and power) of the test.

Simulation Results

data and PSF

result w/ prior I

prior II