# Computing pulse phase and pulse frequency using pulse period ephemeris

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## 1 Problem

In a temporal analysis for pulsar studies in the high-energy astrophysics, it is commonly requested to compute a pulse phase, a pulse frequency, and time derivatives of pulse frequency at a time of interest, such as a photon arrival time, in order to precisely follow rotational motions of a pulsar whose rotation period varies in time. When a temporal variation of pulse period is well-represented by a relatively simple function of time, those values can be directly computed with elementary functions. In this article, explicit functional forms of a pulse phase, a pulse frequency, and time derivatives of pulse frequency at an arbitrary time are presented for cases where a pulse period is represented by a second-order polynomial function of time.

Let $p(t)$ be a pulse period at time $t$, $f(t)$ a pulse frequency at the time, $f_n(t)$ the $n$-th order time derivative of pulse frequency at the time, and $\phi(t)$ a pulse phase at the time, then in general,

$$
\begin{aligned}
f(t) &= \frac{1}{p(t)} && (1)\\
f_n(t) &= \frac{d^n f(t)}{dt^n} && (2)\\
&= \frac{d^n}{dt^n}\left(\frac{1}{p(t)}\right) && (3)\\
\phi(t) &= \phi_0 + \int_{t_0}^{t} f(s)ds && (4)\\
&= \phi_0 + \int_{t_0}^{t} \frac{ds}{p(s)}, && (5)
\end{aligned}
$$

where $t_0$ is an arbitrary reference time, and $\phi_0$ is a pulse phase at time $t_0$. Since $p(t)$ is a second-order polynomial function of time, it can be parameterized as

$$p(t) = p_0 + p_1(t - t_0) + \frac{p_2}{2}(t - t_0)^2. \tag{6}$$

The following sections show the explicit expressions of $f(t)$, $f_n(t)$, and $\phi_0(t)$ with the time $t$ and the period parameters $p_0$, $p_1$, and $p_2$.

# 2 Pulse frequency

Combining Eqs. 1 and 6, one obtains

$$f(t) = \frac{1}{p_0 + p_1(t - t_0) + \frac{p_2}{2}(t - t_0)^2}. \tag{7}$$

# 3 Time derivatives of pulse frequency

Using the proposition in Section A.1, one obtains

$$f_n(t) = \sum_{k=0}^{[n/2]} C_{n,k}\,(n-k)!\left(-\frac{1}{q_0}\right)^{n-k} \frac{{q_1}^{n-2k}\,{p_2}^{k}}{q_0} \tag{8}$$

for $n > 0$, where $C_{n,k}$ are defined by Eqs. 29, 30, and 31, and

$$\begin{aligned} q_0 &= p_0 + p_1(t - t_0) + \frac{p_2}{2}(t - t_0)^2 && (9) \\ q_1 &= p_1 + p_2(t - t_0). && (10) \end{aligned}$$

# 4 Pulse phase

Combining Eqs. 5 and 6, one obtains

$$\begin{aligned} \phi(t) &= \phi_0 + \int_{t_0}^{t} \frac{ds}{p_0 + p_1(s - t_0) + \frac{p_2}{2}(s - t_0)^2} && (11) \\ &= \phi_0 + \int_{0}^{t-t_0} \frac{dx}{\frac{p_2}{2}x^2 + p_1 x + p_0} && (12) \end{aligned}$$

by substituting $s$ with $x = s - t_0$. Letting $a = \frac{p_2}{2}$, $b = p_1$, and $c = p_0$ in the formulas in Section B gives explicit functional forms of $\phi(t)$ as follows.

1. For $p_0 \neq 0$ and $p_1 = p_2 = 0$,

$$\phi(t) = \phi_0 + \frac{t - t_0}{p_0}. \tag{13}$$

2. For $p_1 \neq 0$ and $p_2 = 0$,

$$\phi(t) = \phi_0 + \frac{1}{p_1} \log\left(\frac{p_1(t - t_0)}{p_0} + 1\right) \tag{14}$$

under the condition

$$p_0\{p_0 + p_1(t - t_0)\} > 0. \tag{15}$$

3. For $p_2 \neq 0$ and $2p_0p_2 > {p_1}^2$,

$$\phi(t) = \phi_0 + \frac{2}{\sqrt{2p_0p_2 - {p_1}^2}}(\arctan u_1 - \arctan u_0). \tag{16}$$

where $u_0$ and $u_1$ are defined by

$$u_0 \equiv \frac{p_1}{\sqrt{2p_0p_2 - {p_1}^2}} \tag{17}$$

$$u_1 \equiv \frac{p_1 + p_2(t - t_0)}{\sqrt{2p_0p_2 - {p_1}^2}}. \tag{18}$$

If both of $u_0$ and $u_1$ are either very large or very small, use the following formula instead, in order to minimize a loss of significant digits. See Section B.3 for details.

$$\phi(t) = \phi_0 + \frac{2}{\sqrt{2p_0p_2 - {p_1}^2}} \arctan \frac{(t - t_0)\sqrt{2p_0p_2 - {p_1}^2}}{2p_0 + p_1(t - t_0)}. \tag{19}$$

4. For $p_2 \neq 0$ and $2p_0p_2 = {p_1}^2$,

$$\phi(t) = \phi_0 + \frac{2p_2(t - t_0)}{p_1\{p_1 + p_2(t - t_0)\}}. \tag{20}$$

under the condition

$$p_1\{p_1 + p_2(t - t_0)\} > 0. \tag{21}$$

5. For $p_2 \neq 0$ and $2p_0p_2 < {p_1}^2$,

$$\phi(t) = \phi_0 + \frac{1}{\sqrt{{p_1}^2 - 2p_0p_2}} \log \left| \frac{2p_0 + (p_1 + \sqrt{{p_1}^2 - 2p_0p_2})(t - t_0)}{2p_0 + (p_1 - \sqrt{{p_1}^2 - 2p_0p_2})(t - t_0)} \right|. \tag{22}$$

under the condition

$$x_+\{x_+ - (t - t_0)\} > 0 \quad \text{and} \quad x_-\{x_- - (t - t_0)\} > 0 \tag{23}$$

where $x_\pm$ is defined by

$$x_\pm \equiv -\frac{p_1 \pm \sqrt{{p_1}^2 - 2p_0p_2}}{p_2}. \tag{24}$$

# A Derivatives of a composite function with a second-order polynomial

In this section, a proposition[1] provides an analytic expression of the derivative of an arbitrary order of a composite of two functions, one of which is a second-order polynomial, and another an arbitrary function. Several examples of the proposition are also shown. A proof of the proposition is attached at the end of this section.

## A.1 Proposition

Let $f(y)$ be a differentiable function, $g(x)$ a second-order polynomial (i.e., $g(x) = ax^2 + bx + c$), and the composite of them $F(x) = f(g(x))$. For any integer $n > 0$, using the following notation,

$$F_n \equiv \frac{d^n F(x)}{dx^n} \tag{25}$$

$$f_n \equiv \left[ \frac{d^n f(y)}{dy^n} \right]_{y=g(x)} \tag{26}$$

$$g_n \equiv \frac{d^n g(x)}{dx^n} \tag{27}$$

[1] It is a special case of Fiaà di Bruno's formula.

the $n$-th order derivative of $F(x)$ is given by

$$F_n = \sum_{k=0}^{[n/2]} C_{n,k}\, f_{n-k}\, {g_1}^{n-2k}\, {g_2}^{k}, \tag{28}$$

where $[x]$ is the largest integer less than or equal to a real number $x$ (the floor function), and $C_{m,l}$ integer coefficients defined for integers $m$ and $l$ that satisfy $m > 0$ and $[m/2] \geq l \geq 0$. The coefficients $C_{m,l}$ for $m > 0$ and $[m/2] \geq l > 0$ are defined by

$$C_{m,l} \equiv C_{m-1,l} + (m - 2l + 1)C_{m-1,l-1} \tag{29}$$

$$C_{2l-1,l} \equiv 0 \tag{30}$$

and those for $m > 0$ and $l = 0$ by

$$C_{m,0} \equiv 1. \tag{31}$$

## A.2 Example

Using Eqs. 29, 30, and 31, all the coefficients $C_{m,l}$ for $0 < m \leq 6$ are obtained as the following.

$$\begin{array}{ll}
\text{For} \quad m = 1, & C_{1,0} = 1 \\
& C_{1,1} = 0 \\
\text{For} \quad m = 2, & C_{2,0} = 1 \\
& C_{2,1} = C_{1,1} + C_{1,0} = 1 \\
\text{For} \quad m = 3, & C_{3,0} = 1 \\
& C_{3,1} = C_{2,1} + 2C_{2,0} = 3 \\
& C_{3,2} = 0 \\
\text{For} \quad m = 4, & C_{4,0} = 1 \\
& C_{4,1} = C_{3,1} + 3C_{3,0} = 6 \\
& C_{4,2} = C_{3,2} + C_{3,1} = 3 \\
\text{For} \quad m = 5, & C_{5,0} = 1 \\
& C_{5,1} = C_{4,1} + 4C_{4,0} = 10 \\
& C_{5,2} = C_{4,2} + 2C_{4,1} = 15 \\
& C_{5,3} = 0
\end{array}$$

$$\begin{aligned} \text{For} \quad m = 6, \quad & C_{6,0} = 1 \\ & C_{6,1} = C_{5,1} + 5C_{5,0} = 15 \\ & C_{6,2} = C_{5,2} + 3C_{5,1} = 45 \\ & C_{6,3} = C_{5,3} + C_{5,2} = 15 \end{aligned}$$

Using those coefficients obtained above, the derivatives of $F(x)$ are obtained up to the 6th order as the following.

$$\begin{aligned} F_1(x) &= f_1 g_1 \\ F_2(x) &= f_2 {g_1}^2 + f_1 g_2 \\ F_3(x) &= f_3 {g_1}^3 + 3 f_2 g_1 g_2 \\ F_4(x) &= f_4 {g_1}^4 + 6 f_3 {g_1}^2 g_2 + 3 f_2 {g_2}^2 \\ F_5(x) &= f_5 {g_1}^5 + 10 f_4 {g_1}^3 g_2 + 15 f_3 g_1 {g_2}^2 \\ F_6(x) &= f_6 {g_1}^6 + 15 f_5 {g_1}^4 g_2 + 45 f_4 {g_1}^2 {g_2}^2 + 15 f_3 {g_2}^3 \end{aligned}$$

## A.3 Proof

The proposition is proved below by mathematical induction. Let $G_n(x)$ be the right-hand side of Eq. 28, i.e.,

$$G_n = \sum_{k=0}^{[n/2]} C_{n,k}\, f_{n-k}\, {g_1}^{n-2k}\, {g_2}^k \tag{32}$$

For $n = 1$, $F_1(x)$ is derived by the chain rule as

$$\begin{aligned} F_1(x) &= \frac{df(g(x))}{dx} && (33) \\ &= \left[\frac{df(y)}{dy}\right]_{y=g(x)} \cdot \frac{dg(x)}{dx} && (34) \\ &= f_1 g_1. && (35) \end{aligned}$$

Combined with eqs. 29, 30, and 31, the definition of $G_n(x)$ gives $G_1(x) = f_1 g_1$. Therefore, $F_1(x) = G_1(x)$. Assume $F_n(x) = G_n(x)$ is true for an integer $n \geq 1$. Then,

$$F_{n+1}(x) = \frac{dF_n(x)}{dx} \tag{36}$$

$$
\begin{aligned}
&= \frac{dG_n(x)}{dx} & (37)\\
&= \frac{d}{dx}\sum_{k=0}^{[n/2]} C_{n,k}\, f_{n-k}\, {g_1}^{n-2k}\, {g_2}^{k}, & (38)
\end{aligned}
$$

Since $g(x)$ is a second-order polynomial of $x$,

$$
\frac{dg_2}{dx} = 0. \qquad (39)
$$

Also, the chain rule gives

$$
\frac{df_{n-k}}{dx} = f_{n-k+1}\, g_1. \qquad (40)
$$

Hence,

$$
\begin{aligned}
F_{n+1}(x) &= \sum_{k=0}^{[n/2]} C_{n,k} \Big\{ f_{n-k+1}\, {g_1}^{n-2k+1} \\
&\quad + (n-2k) f_{n-k}\, {g_1}^{n-2k-1}\, g_2 \Big\}\, {g_2}^{k} & (41)\\
&= f_{n+1}\, {g_1}^{n+1} + \sum_{k=1}^{[n/2]} C_{n,k}\, f_{n-k+1}\, {g_1}^{n-2k+1}\, {g_2}^{k} \\
&\quad + \sum_{k=0}^{[n/2]-1} (n-2k) C_{n,k}\, f_{n-k}\, {g_1}^{n-2k-1}\, {g_2}^{k+1} + R_n & (42)
\end{aligned}
$$

where $R_n$ is defined by

$$
R_n \equiv C_{n,[n/2]}\, f_{n-[n/2]}\, (n-2[n/2]) {g_1}^{n-2[n/2]-1}\, {g_2}^{[n/2]+1}. \qquad (43)
$$

With simple manipulations, one derives

$$
\begin{aligned}
F_{n+1}(x) &= f_{n+1}\, {g_1}^{n+1} + \sum_{k=1}^{[n/2]} C_{n,k}\, f_{n-k+1}\, {g_1}^{n-2k+1}\, {g_2}^{k} \\
&\quad + \sum_{k=1}^{[n/2]} (n-2k+2) C_{n,k-1}\, f_{n-k+1}\, {g_1}^{n-2k+1}\, {g_2}^{k} + R_n & (44)\\
&= f_{n+1}\, {g_1}^{n+1} \\
&\quad + \sum_{k=1}^{[n/2]} \{ C_{n,k} + (n-2k+2) C_{n,k-1} \}\, f_{n-k+1}\, {g_1}^{n-2k+1}\, {g_2}^{k} \\
&\quad + R_n & (45)
\end{aligned}
$$

From Eq. 29, one obtains

$$C_{n,k} + (n-2k+2)C_{n,k-1} = C_{n+1,k}, \tag{46}$$

therefore,

$$\begin{aligned} F_{n+1}(x) &= f_{n+1}\, {g_1}^{n+1} \\ &\quad + \sum_{k=1}^{[n/2]} C_{n+1,k}\, f_{n-k+1}\, {g_1}^{n-2k+1}\, {g_2}^{k} \\ &\quad + R_n & (47) \\ &= \sum_{k=0}^{[n/2]} C_{n+1,k}\, f_{n-k+1}\, {g_1}^{n-2k+1}\, {g_2}^{k} + R_n & (48) \end{aligned}$$

For an even $n$, $[n/2] = n/2$, hence $R_n = 0$. Also, $[n/2] = [(n+1)/2]$. Therefore, one obtains

$$F_{n+1}(x) = \sum_{k=0}^{[(n+1)/2]} C_{n+1,k}\, f_{n-k+1}\, {g_1}^{n-2k+1}\, {g_2}^{k} \tag{49}$$

For an odd $n$, $[n/2] = (n-1)/2$ and $[n/2] = [(n+1)/2] - 1$. Combined with Eqs. 29 and 30, one obtains

$$\begin{aligned} C_{n+1,[(n+1)/2]} &= C_{n,[(n+1)/2]} + C_{n,[(n+1)/2]-1} & (50) \\ &= C_{n,(n+1)/2} + C_{n,[(n+1)/2]-1} & (51) \\ &= C_{n,[(n+1)/2]-1}. & (52) \end{aligned}$$

Therefore,

$$\begin{aligned} R_n &= C_{n,[(n+1)/2]-1}\, f_{n-[(n+1)/2]+1}\, {g_1}^{n-2[(n+1)/2]+1}\, {g_2}^{[(n+1)/2]} & (53) \\ &= C_{n+1,[(n+1)/2]}\, f_{n-[(n+1)/2]+1}\, {g_1}^{n-2[(n+1)/2]+1}\, {g_2}^{[(n+1)/2]} & (54) \\ &= \left[C_{n+1,k}\, f_{n-k+1}\, {g_1}^{n-2k+1}\, {g_2}^{k}\right]_{k=[(n+1)/2]}, & (55) \end{aligned}$$

and finally,

$$\begin{aligned} F_{n+1}(x) &= \sum_{k=0}^{[(n+1)/2]-1} C_{n+1,k}\, f_{n-k+1}\, {g_1}^{n-2k+1}\, {g_2}^{k} \\ &\quad + \left[C_{n+1,k}\, f_{n-k+1}\, {g_1}^{n-2k+1}\, {g_2}^{k}\right]_{k=[(n+1)/2]} & (56) \\ &= \sum_{k=0}^{[(n+1)/2]} C_{n+1,k}\, f_{n-k+1}\, {g_1}^{n-2k+1}\, {g_2}^{k} & (57) \end{aligned}$$

Eqs. 49 and 57 show Eq. 28 is true for $n+1$ for all cases. Q.E.D.

# B Definite integral of the reciprocal of a second-order polynomial function

In this section, explicit functional forms of the definite integral of the reciprocal of a second-order polynomial is derived. Let $I(x_1)$ be the definite integral of interest defined by

$$I(x_1) \equiv \int_0^{x_1} \frac{dx}{ax^2 + bx + c}. \tag{58}$$

Antiderivatives for the integrand are elementary functions, but the their functional forms depend on the parameter values. Below each of the cases is described separately.

## B.1 Case for $a = b = 0$ and $c \neq 0$

In this case, the antiderivatives are given by

$$\int \frac{dx}{c} = \frac{x}{c}. \tag{59}$$

Therefore, one obtains

$$I(x_1) = \frac{x_1}{c}. \tag{60}$$

## B.2 Case for $a = 0$ and $b \neq 0$

In this case, the antiderivatives are given by

$$\int \frac{dx}{bx + c} = \frac{1}{b} \log |bx + c|. \tag{61}$$

Therefore, one obtains

$$\begin{aligned} I(x_1) &= \frac{1}{b} \left[\log |bx + c|\right]_0^{x_1} & (62) \\ &= \frac{1}{b} \log \left| \frac{bx_1}{c} + 1 \right|. & (63) \end{aligned}$$

Note that Eq. 67 holds only if $bx + c \neq 0$ at all $x$ in the range of integration. Since there is only one value of $x$ which satisfies $bx + c = 0$, the condition can be expressed by

$$x_*(x_* - x_1) > 0 \tag{64}$$

where $x_*$ is the solution of the equation, namely,

$$x_* \equiv -\frac{c}{b}. \tag{65}$$

Since $b \neq 0$, the condition is equivalent to

$$c(bx_1 + c) > 0. \tag{66}$$

Since the argument of the logarithmic function in Eq. 63 is always positive under the condition, Eq. 63 can be transformed to

$$I(x_1) = \frac{1}{b}\log\left(\frac{bx_1}{c} + 1\right). \tag{67}$$

## B.3 Case for $a \neq 0$ and $4ac > b^2$

In this case, the antiderivatives are given by

$$\int \frac{dx}{ax^2 + bx + c} = \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}}. \tag{68}$$

Therefore, one obtains

$$I(x_1) = \frac{2}{\sqrt{4ac - b^2}} \left[\arctan \frac{2ax + b}{\sqrt{4ac - b^2}}\right]_0^{x_1} \tag{69}$$

$$= \frac{2(\theta_1 - \theta_0)}{\sqrt{4ac - b^2}} \tag{70}$$

where $\theta_0$ and $\theta_1$ are defined by

$$\theta_0 \equiv \arctan \frac{b}{\sqrt{4ac - b^2}} \tag{71}$$

$$\theta_1 \equiv \arctan \frac{2ax_1 + b}{\sqrt{4ac - b^2}}. \tag{72}$$

A special attention needs to be paid in computing the right-hand side of Eq. 70 in a computer system for cases where both of $|\tan\theta_0|$ and $|\tan\theta_1|$ are large. For $|x| \gg 1$, $|\arctan x| > 1$ and $x^2 + 1 \gg |x|$, hence,

$$\left|\frac{d}{dx}\arctan x\right| = \frac{1}{x^2 + 1} \ll \left|\frac{\arctan x}{x}\right|. \tag{73}$$

This means, a fractional change in a value of the arctangent function is much smaller than that in an argument value when the argument value is large. As a result, a direct computation of $(\theta_1 - \theta_0)$ causes a significant loss of digits. To mitigate such a loss of digits, one can further transform Eq. 70.

From the sum formulas the trigonometric functions, one obtains

$$\tan(\theta_1 - \theta_0) = \frac{\tan\theta_1 - \tan\theta_0}{1 + \tan\theta_1 \tan\theta_0} \tag{74}$$

$$= \frac{x_1\sqrt{4ac - b^2}}{bx_1 + 2c}. \tag{75}$$

For $\tan\theta_0 \gg 1$ and $\tan\theta_1 \gg 1$, $\theta_0 > \frac{\pi}{4}$ and $\theta_1 > \frac{\pi}{4}$. Also, by definition, $\theta_0 < \frac{\pi}{2}$ and $\theta_1 < \frac{\pi}{2}$. Therefore, one obtains $-\frac{\pi}{4} < \theta_1 - \theta_0 < \frac{\pi}{4}$, then

$$\theta_1 - \theta_0 = \arctan\frac{x_1\sqrt{4ac - b^2}}{bx_1 + 2c}. \tag{76}$$

Similarly, for $\tan\theta_0 \ll -1$ and $\tan\theta_1 \ll -1$, it is trivial that $-\frac{\pi}{4} < \theta_1 - \theta_0 < \frac{\pi}{4}$, then Eq. 76 holds. Combining Eqs. 70 and 76, one obtains

$$I(x_1) = \frac{2}{\sqrt{4ac - b^2}} \arctan\frac{x_1\sqrt{4ac - b^2}}{bx_1 + 2c}. \tag{77}$$

for cases with $\tan\theta_0 \gg 1$ and $\tan\theta_1 \gg 1$ and those with $\tan\theta_0 \ll -1$ and $\tan\theta_1 \ll -1$.

With Eq. 77, one can avoid a loss of significant digits in a direct computation of $(\theta_1 - \theta_0)$ in Eq. 70. Eq. 77, on the other hand, is not applicable to cases with $|\theta_1 - \theta_0| \geq \frac{\pi}{2}$ because values of the arctangent function range only between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. In other words, one can benefit from using Eq. 77 to compute $I(x_1)$ if both of $\tan\theta_0$ and $\tan\theta_1$ are either very large or very small, and Eq. 70 covers all other cases.

### B.4 Case for $a \neq 0$ and $4ac = b^2$

In this case, the antiderivatives are given by

$$\int \frac{dx}{ax^2 + bx + c} = -\frac{2}{2ax + b}. \tag{78}$$

Therefore, one obtains

$$
\begin{aligned}
I(x_1) &= \left[-\frac{2}{2ax+b}\right]_0^{x_1} && (79)\\
&= \frac{4ax_1}{b(2ax_1+b)}. && (80)
\end{aligned}
$$

Note that Eq. 80 holds only if $ax^2+bx+c \neq 0$ at all $x$ in the range of integration. Since there is only one value of $x$ which satisfies $ax^2+bx+c=0$, the condition can be expressed by

$$
x_*(x_* - x_1) > 0 \tag{81}
$$

where $x_*$ is the solution of the equation, namely,

$$
x_* \equiv -\frac{b}{2a}. \tag{82}
$$

Since $a \neq 0$, the condition is equivalent to

$$
b(2ax_1+b) > 0. \tag{83}
$$

## B.5 Case for $a \neq 0$ and $4ac < b^2$

In this case, the antiderivatives are given by

$$
\int \frac{dx}{ax^2+bx+c} = \frac{1}{\sqrt{b^2-4ac}} \log\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| \tag{84}
$$

Therefore, one obtains

$$
\begin{aligned}
I(x_1) &= \frac{1}{\sqrt{b^2-4ac}}\left[\log\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|\right]_0^{x_1} && (85)\\
&= \frac{1}{\sqrt{b^2-4ac}} \log\left|\frac{2ax_1+b-\sqrt{b^2-4ac}}{2ax_1+b+\sqrt{b^2-4ac}} \cdot \frac{b+\sqrt{b^2-4ac}}{b-\sqrt{b^2-4ac}}\right| && (86)\\
&= \frac{1}{\sqrt{b^2-4ac}} \log\left|\frac{x_1(b+\sqrt{b^2-4ac})+2c}{x_1(b-\sqrt{b^2-4ac})+2c}\right| && (87)
\end{aligned}
$$

Note that Eq. 87 holds only if $ax^2+bx+c \neq 0$ at all $x$ in the range of integration. Since there are two values of $x$ which satisfy $ax^2+bx+c=0$, the condition can be expressed by

$$
x_+(x_+ - x_1) > 0 \quad \text{and} \quad x_-(x_- - x_1) > 0 \tag{88}
$$

where $x_{\pm}$ are the solutions of the equation, namely,

$$x_{\pm} \equiv -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{89}$$