A Little History, but... in Astronomy its Resolution, Resolution, Resolution

Bill Atwood
SCIPP/UCSC
Gamma Ray Pair Conversion

Energy loss mechanisms

Fig. 2: Photon cross-section $\sigma$ in lead as a function of photon energy. The intensity of photons can be expressed as $I = I_0 \exp(-\alpha x)$, where $x$ is the path length in radiation lengths. (Review of Particle Properties, April 1980 edition).

Energy

Splitting Function

Opening Angle

$\theta_{\text{Open}} \approx \frac{4m_e}{E_\gamma}$

At 100 MeV

$\theta_{\text{Open}} \sim 1^\circ$

From Rossi, High Energy Particles, 1952

High Electric Field

Pair

$e^+ e^-$

Position Measuring Detectors

Measured Track co-ordinates

Plastic Scintillation counters to veto entering charged particles

Total Absorption Calorimeter to measure gamma ray energy

Z=74 Tungsten Conversion Foils

QED Process

$\gamma$
Previous Satellite Detectors

- **1967-1968, OSO-3** Detected Milky Way as an extended $\gamma$-ray source
  - 621 $\gamma$-rays
- **1972-1973, SAS-2**, ~8,000 $\gamma$-rays
- **1975-1982, COS-B**
  orbit resulted in a large and variable background of charged particles
  - ~200,000 $\gamma$-rays
- **1991-2000, EGRET**
  Large effective area, good PSF, long mission life, excellent background rejection
  - $>1.4 \times 10^6$ $\gamma$-rays
Evolution of GLAST

- April, 1991 CGRO (with EGRET on board) Shuttle Launch
- May, 1992 NASA SR & T Proposal Cycle

1. Select the Technologies

Large area SSD systems and CsI Calorimeters resulted from SSC R&D

2. Make it Modular

Another lesson learned in the 1980's: monolithic detectors are inferior to Segmented detectors

3. Pick the Rocket

Delta II (launch of GP-B)

Cheap, reliable Communication satellite launch vehicle

4. Fill-it-up!

Rocket Payload Fairing

Diameter sets transverse size

Lift capacity to LEO sets depth of Calorimeter

Original GISMO 1 Event Displays from the first GLAST simulations
Technology of Choice: Solid State

Silicon Strip Detector Principle

Custom Integrated Circuits (ASICs)

Low-noise, Low-power Amplifier/Discriminator (S/N typically > 20)

GLAST has 884736 channels. Total Tracker Power = 160 Watts!
Pair Conversion Telescope

Expanded view of converter-tracker:

\[ \gamma + Z \rightarrow e^+ + e^- + Z_{\text{recoil}} \]

At 100 MeV, opening angle ~ 20 mrad

All detectors have some dead area: if isolated, can trim converter to cover only active area; if distributed, conversions above or near dead region contribute tails to PSF unless detailed and efficient algorithms can ID and remove such events.

Low energy PSF completely dominated by multiple scattering effects:
\[ \theta_0 \sim 2.9 \text{ mrad} / E[\text{GeV}] \]
(scales as \( (x_0)^{1/2} \))

High energy PSF set by hit resolution/plane spacing:
\[ \theta_D \sim 1.8 \text{ mrad} \]

At higher energies, more planes contribute information:

<table>
<thead>
<tr>
<th>Energy</th>
<th># significant planes</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 MeV</td>
<td>2</td>
</tr>
<tr>
<td>1 GeV</td>
<td>~5</td>
</tr>
<tr>
<td>10 GeV</td>
<td>&gt;10</td>
</tr>
</tbody>
</table>
**Tracker Design and Analysis**

### Pair Conversion

- $\gamma$ converts $\frac{1}{2}$ through radiator

### Telescope Layout

- Close spacing of Radiators to SSDs minimizes multiple scattering effects
  
  \[ \chi_{\text{eff}}(\text{MS}) = \frac{\chi_w}{2} + 2 \cdot \chi_{\text{SSD}} = 0.025 \]

- Trim Radiator tiles to match active SSD area

- Plane-to-plane spacing and SSD strip pitch sets meas. precision limit

### Kalman Tracking/Fitting

- Track parameters (position, angles, error matrix) at a plane

- Propagation of parameters

- Multiple Scattering -- depends on energy!

- Measurement with error

- New parameters at next plane

### Angular Resolution Parameters

- $\delta \theta_{\text{MS}}(100\text{MeV}) \approx 38\text{mrad}$

- $\delta \theta_{\text{MS}}(\text{Space}) \approx \sqrt{2} \cdot 38\text{mrad} = 54\text{mrad} = 3.1^\circ$

- $\delta \theta_{\text{Det}} = \frac{\sigma_{\text{SSD}}}{d} \cdot \sqrt{2} = \frac{\text{Pitch}/\sqrt{12}}{d} \cdot \sqrt{2}$

- $\delta \theta_{\text{Det}} = \frac{228\mu\text{m}}{32.9\text{mm} \cdot \sqrt{6}} = 2.8\text{mrad} = 0.16^\circ$

- Multiple Scattering

\[ \theta_o = \frac{14\text{mrad} - \text{GeV}}{p\beta} \cdot \sqrt{\chi_{\text{Eff}}} \approx \frac{14\text{mrad} - \text{GeV}}{0.5E_{\gamma}} \cdot \sqrt{\chi_{\text{Eff}}} \]

### Trade Between $A_{\text{eff}}$ & PSF

- $N_{\gamma} \propto \chi_{\text{Rad}}$

- $\text{PSF} \propto \sqrt{\chi_{\text{Rad}}}$

- Source Sensitivity $\Rightarrow$ Photon Density

\[ \text{Sens.} \approx \frac{N_{\gamma}}{\text{PSF}^2} \]

- Does'nt depend on $\chi_{\text{Rad}}$!

- 2-Source Separation pushes for thin radiators

- Transient sensitivity pushes for thick radiators

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**Diversion: Review of Covariance**

**Ellipse**

Take a circle – scale the x & y axis:

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

Rotate by \( \theta \):

\[
x \rightarrow x \cos(\theta) + y \sin(\theta) \\
y \rightarrow y \cos(\theta) - x \sin(\theta)
\]

Results:

\[
x^2 \left( \frac{\cos^2(\theta)}{a^2} + \frac{\sin^2(\theta)}{b^2} \right) + 2xy \cos(\theta) \sin(\theta) \left( \frac{1}{a^2} - \frac{1}{b^2} \right) + y^2 \left( \frac{\sin^2(\theta)}{a^2} + \frac{\cos^2(\theta)}{b^2} \right) = 1
\]

Rotations mix \( x \) & \( y \). Major & minor axis plus rotation angle \( \theta \) complete description.

**Error Ellipse described by Covariance Matrix:**

Distance between a point with an error and another point measured in \( \sigma \)'s:

\[
(n\sigma)^2 = r^T C^{-1} r \text{ where } r = (\vec{x} - \vec{x}) \text{ and } \\
C^{-1} = \text{Inverse}(C) = \begin{bmatrix} C_{xx}^{-1} & C_{xy}^{-1} \\ C_{yx}^{-1} & C_{yy}^{-1} \end{bmatrix} \text{ and } C_{xy}^{-1} = C_{yx}^{-1}
\]

Simply weighting the distance by \( 1/\sigma^2 \)
Multiplying it out gives:

\[(n\sigma)^2 = r^T C^{-1} r = (x, y) \begin{bmatrix} C_{xx}^{-1} & C_{xy}^{-1} \\ C_{xy}^{-1} & C_{yy}^{-1} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2C_{xx}^{-1} + 2xyC_{xy}^{-1} + y^2C_{yy}^{-1}\]

Where I take \( \bar{x} = 0 \) without loss of generality.

This is the equation of an ellipse! Specifically for a 1σ error ellipse \((n\sigma = 1)\) we identify:

\[C_{xx}^{-1} = \frac{\cos^2(\theta)}{a^2} + \frac{\sin^2(\theta)}{b^2} \quad C_{yy}^{-1} = \frac{\sin^2(\theta)}{a^2} + \frac{\cos^2(\theta)}{b^2} \quad C_{xy}^{-1} = \sin(\theta)\cos(\theta)(\frac{1}{a^2} - \frac{1}{b^2})\]

and \(C^{-1} = \frac{1}{\det(C)} \begin{bmatrix} C_{yy} & -C_{xy} \\ -C_{xy} & C_{xx} \end{bmatrix}\) where \(\det(C) = (C_{xx}C_{yy} - C_{xy}^2)\)

The correlation coefficient is defined as: \(R^2 = \frac{C_{xy}^2}{C_{xx}C_{yy}}\)

**Summary:** The Covariance Matrix describes an ellipse where the major and minor axis and the rotation angle map directly onto its components!
Covariance - 3

Let the fun begin! To disentangle the two descriptions consider

\[ A = \frac{C_{xy}^{-1}}{C_{xx}^{-1} + C_{yy}^{-1}} = \frac{-C_{xy}}{C_{xx} + C_{yy}} = \frac{\cos(\theta) \sin(\theta) (b^2 - a^2)}{a^2 + b^2} \]

thus \[ A = \frac{\sin(2\theta)}{2} \left( \frac{1 - r^2}{1 + r^2} \right) \] where \( r = \frac{a}{b} \)

Also \( \det(C) \) yields (with a little algebra & trig.):

\[ a \cdot b = \sqrt{\det(C)} = \frac{\text{Area}}{\pi} \]

Now we’re ready to continue the story of GLAST (Fermi-LAT)!
The Kalman filter process is a successive approximation scheme to estimate parameters.

**Simple Example:** 2 parameters - intercept and slope: \( x = x_0 + S_x \cdot z \); \( P = (x_0, S_x) \)

**Errors on parameters** \( x_0 \) & \( S_x \): Covariance Matrix \( C = \begin{pmatrix} C_{xx} & C_{xs} \\ C_{sx} & C_{ss} \end{pmatrix} \)

\( C_{xx} = <(x-x_m)(x-x_m)> \)

**In general** \( C = <(P - P_m)(P-P_m)^T> \)

**Propagation:**

\( x(k+1) = x(k) + S_x(k)(z(k+1)-z(k)) \)

\( P(k+1) = F(\delta z) \cdot P(k) \) where

\( F(\delta z) = \begin{pmatrix} 1 & z(k+1)-z(k) \\ 0 & 1 \end{pmatrix} \)

\( C(k+1) = F(\delta z) \cdot C(k) \cdot F(\delta z)^T + Q(k) \)

**Noise:** \( Q(k) \) (Multiple Scattering)
Form the weighted average of the k+1 measurement and the propagated track model:
Weights given by inverse of Error Matrix: $C^{-1}$

Hit: $x(k+1)$ with errors $V(k+1)$

$$P(k+1) = \frac{C^{-1}(k+1)P(k+1) + V^{-1}(k+1)x(k+1)}{C^{-1}(k+1) + V^{-1}(k+1)}$$

and

$$C(k+1) = (C^{-1}(k+1) + V^{-1}(k+1))^{-1}$$

Now its repeated for the k+2 planes and so - on. This is called FILTERING - each successive step incorporates the knowledge of previous steps as allowed for by the NOISE and the aggregate sum of the previous hits.
How Well does it work?

Really a game of “Do you get out what you put in?”

1 GeV Muons used for testing
Energy for Kalman Fit = MC Energy
Results: Large Tail on $\chi^2$
Solution: Include energy losses in Tracker (Bethe-Bloch)

1 GeV Muons
Dependence on $\cos(\theta)$

Next: $\chi^2$ Depends on Angle

Large Angles $\rightarrow$ too Narrow!!

…Suspect Meas. Errors
Re-Work of Meas. Errors

Gaussian Equivalent $\sigma$ for a Square Distribution

$$\sigma^2 = \int_{-1/2}^{+1/2} x^2 \, dx = \frac{x^3}{3} \bigg|_{-1/2}^{+1/2} = \frac{1}{12} \cdot \text{Width} \quad \text{Hence} \quad \frac{\text{Strip - Pitch}}{\sqrt{12}} = \frac{228 \, \mu m}{\sqrt{12}} = 65.8 \, \mu m$$

Actual "hits" on tracks are in general Clusters of Strips. Naively expect $\sigma_{\text{Cluster}} = \frac{\text{Cluster Width}}{\sqrt{12}}$

**But...**

Suggests $\sigma_{\text{Cluster}} = \frac{\text{Strip Pitch}}{\sqrt{12}}$ (!)

Can move track left-right by at most 1 strip pitch!

**θ < 45**

**θ > 45**
Success! $\chi^2$ Distributions – Text-Book!

$$\cos(\theta) = -1$$

$$\langle N_{\text{hits}} \rangle = 36$$
$$\langle \chi^2 \rangle = 1.05$$

$$\langle \sigma_{\text{FIT}} \rangle = 3.4 \text{ mrad}$$

$$-1 < \cos(\theta) < 0$$

$$\langle N_{\text{hits}} \rangle = 22$$
$$\langle \chi^2 \rangle = 1.06$$

$$\langle \sigma_{\text{FIT}} \rangle = 4.0 \text{ mrad}$$

Notice the Binning Effects?
**Vertexing: Two Problems**

1. **Z Location of the Vertex**

   - **Preferred Solution**
     If 2 Tracks share the same first hit and the Cluster Size is no more then 2 Strips and the first hit directly proceed a W radiator:
     \[ Z_{VTX} = \text{middle of W Radiator prior to first hit} \]

   - **Next Best Solution**
     If DOCA location of 2 tracks lies before first hits but is after the next layer up:
     \[ Z_{VTX} = \text{DOCA} - Z \]

   - **All Other Cases**
     Put Vertex at Z location of start of the 1st Track
2. Covariant Averaging of Tracks

Multivariate Averaging: \[ P_{\text{Pair}} = \frac{C_1^{-1} \cdot P_1 + C_2^{-1} \cdot P_2}{C_1^{-1} + C_2^{-1}} \]

\[ P_{\text{Pair}} = (C_1^{-1} + C_2^{-1})^{-1} \cdot (C_1^{-1} \cdot P_1 + C_2^{-1} \cdot P_2) \]

\[ C_{\text{Pair}} = (C_1^{-1} + C_2^{-1})^{-1} \]

where \( P_i \) are the parameter vectors of the combination(Pair) and tracks \( (P_1 \text{ and } P_2) \) and \( C_i \) are the covariance matrices

And

\[ \chi^2 = (P_1 - P_{\text{Pair}})^T C_{\text{Res}_1}^{-1} (P_1 - P_{\text{Pair}}) + (P_2 - P_{\text{Pair}})^T C_{\text{Res}_2}^{-1} (P_2 - P_{\text{Pair}}) \]

where

\[ C_{\text{Res}_{1,2}} = C_{1,2} - C_{\text{Pair}} \]

The parameter vectors \( P \) are \( (x, S_x, y, S_y) \)
Neutral Energy Concept

Sometimes at the start of the shower the charge pair does not well reflect the direction of the incoming photon.

Bremstrahlung can cause much (most) of the energy to windup in photons.

The Calorimeter centroid is a measure of where these photons impact the calorimeter.

A "Neutral Energy" direction can be inferred by connecting the found vertex with the Cal. Centroid.

One can determine the covariant error matrix for this inferred direction by using the errors on the centroid location.

By having an imaging calorimeter, Fermi-LAT is the first Gamma Ray instrument able to do this!
Where does the Charged Solution go Wrong?

At energies < 1 GeV only the first 2 Tracker Hits determine the direction

Internal and External Brems. distort direction
This is in addition to multiple scattering

Expect effect to be more severe in 18% radiators
(Thick - 18%, Thin - 3%)

Brems. in 2nd and lower decks doesn't effect direction

Due to Internal Brems.
ratio of effected events
(Thin : Thick Decks)
will be < ratio of Rad. Lens.

Here's the rub...

Internal Radiator

Real or External Radiator

Tungsten Radiator
Is this Effect Present in the Monte Carlo?

Define $\theta_{\text{NC}}$ to be the angle between The VTX (Charged Tracks) Direction and the Neutral Energy Direction.

Note that the tail of the PSF is strongly correlated with $\theta_{\text{NC}}$ with unit slope!

Also this correlated tail is more pronounced in the Thick Radiators (as expected!)

And... the tail extends from one side of the FoV to the other!

PSF - 95%  PSF - 99%
**Neut. Energy Implementation**

**First Task:** Determine what the errors are as a function of energy.
Naively one expects the location error to $\sim R_{\text{Moliere}}/\sqrt{E}$

Run $\gamma$s [100 MeV, 10 GeV] in a patch $(x,y) = ([100,150], [100,150])$
at normal incidence ($\hat{e}_z = -1$)

**Second Task:** Combine Neutral energy direction covariantly with Charged Soln.

$$P_{\text{SUM}} = C_{\text{SUM}} (C^{-1}_{\text{Charged}} P_{\text{Charged}} + C^{-1}_{\text{Neutral}} P_{\text{Neutral}})$$

where $P_i$ and $C_i$ are the 4-element parameter vectors and the 4x4 Covariance Matrices and $C_{\text{SUM}} = (C^{-1}_{\text{Charged}} + C^{-1}_{\text{Neutral}})^{-1}$
**All Gamma Results**

**Event Selection:** \( \text{CTBClassLevel} > 0 \) \& \( \text{CTBCORE} > .1 \)

**Events Using Neutral Energy Solution**

**Comments**
- 44% events use the Neutral Energy Solution
- The effectiveness of using Neutral Energy increases with increasing energy
- The far tails on the PSF are reigned in
Making Covariance Usable

The Chicken and Egg problem

In this case the “Chickens” are the tracks and the “Eggs” are the energy determinations for the tracks.

Tracking needs an estimation of the event energy.

Energy reconstruction benefits greatly from knowing about the tracks.

Trouble: Final Energy can be factors off from the energy used in the 2nd Kalman Fit
Fixing the Energy Problem

Elements of the Cov. Matrix scales as $1/E^2$ below $\sim 10$ GeV (Mult. Scat. Dominated)

Suggests $C_{\text{FINAL}} \approx C_{\text{FIT}} \cdot \left( \frac{E_{\text{FINAL}}}{E_{\text{FIT}}} \right)^2$

Best agreement found with

$C_{\text{FINAL}} \approx C_{\text{FIT}} \cdot \left( \frac{E_{\text{FINAL}}}{E_{\text{FIT}}} \right)^{1.6}$

The power 1.6 is consistent with PSF behavior ($1/E^{0.8}$) (Tyrel Johnson, 2007)

Correct Soln.: Iterate Fit with Final Energy

Find iteration ONLY affects Cov. Matrix, NOT the Parameters!

Transformation of the Track Covariance Matrix to Sky Coordinates
R.P. Johnson  September 19, 2007

The Kalman fit gives us a covariance matrix in terms of the slope and intercept in each projection. Only the two slopes are relevant to the photon direction, so the first step is to reduce the error matrix from $4 \times 4$ to $2 \times 2$, simply by removing the rows and columns related to the intercepts. Let's call the two slopes $s_1$ and $s_2$. The associated $2 \times 2$ covariance matrix is $\sigma^2_{ss}$.

Let $\hat{\theta}$ be the unit vector (direction cosines) denoting the downward (per LAT convention) photon direction in the LAT coordinate system and $\hat{\psi}$ the corresponding unit vector in the sky coordinates (galactic coordinates, for example). Finally, let $\Theta = (\ell, b)$ be the photon longitude and latitude in the sky coordinates.

The transformation from $\hat{\theta}$ to $\hat{\psi}$ is linear and can be represented by a $3 \times 3$ orthogonal matrix $R$ that includes a space inversion (i.e. negative determinant). The transformations from $s$ to $\hat{\theta}$ and from $\hat{\psi}$ to $\Theta$ are nonlinear, but for purposes of transforming the covariance matrix, all we need are the first-derivative matrices. Let $A$ represent the $3 \times 2$ derivative matrix for the transformation $s \rightarrow \hat{\theta}$ and $B$ the $2 \times 3$ derivative matrix for $\hat{\psi} \rightarrow \Theta$. The $2 \times 2$ matrix for the overall transformation $s \rightarrow \Theta$ then is $M = BRA$, and the covariance matrix transforms as

$$
\sigma^2_{ss} = M \sigma^2_{ss} M^T.
$$

The transformation equations corresponding to $A$ can be written

$$
\begin{align*}
\epsilon_1 &= -[1 + \epsilon_1^2 + \epsilon_2^2]^{-1/2} \\
\epsilon_2 &= \epsilon_1 \epsilon_2 \epsilon_1 \\
\epsilon_3 &= \epsilon_1 \epsilon_2 \\
\end{align*}
$$

and the matrix $A = \partial \hat{\theta}/\partial s$ can be written in terms of the direction cosines $\hat{\psi}$ as

$$
A = \begin{pmatrix}
-\hat{\psi}_x \epsilon_2 + \hat{\psi}_y \epsilon_1 & -\hat{\psi}_x \epsilon_1 - \hat{\psi}_y \epsilon_2 \\
\hat{\psi}_x \epsilon_1 - \hat{\psi}_y \epsilon_2 & \hat{\psi}_x \epsilon_2 + \hat{\psi}_y \epsilon_1
\end{pmatrix}.
$$

The transformation equations corresponding to $B$ can be written

$$
\begin{align*}
\ell &= \tan^{-1}(\hat{\psi}_y / \hat{\psi}_x) \\
b &= \sin^{-1} \hat{\psi}_z
\end{align*}
$$

and the matrix $B = \partial \Theta / \partial \hat{\psi}$ is

$$
B = \begin{pmatrix}
\hat{\psi}_x / \sqrt{\hat{\psi}_x^2 + \hat{\psi}_y^2} & \hat{\psi}_y / \sqrt{\hat{\psi}_x^2 + \hat{\psi}_y^2} & 0 \\
0 & 0 & \sqrt{\hat{\psi}_x^2 + \hat{\psi}_y^2}
\end{pmatrix}.
$$

These equations are singular at the galactic pole ($\hat{\psi}_z = 1$), which has to be protected, and also at $\hat{\psi}_z = 0$, which should never occur, given that the Tracker cannot find horizontal tracks.
Neronov & Collaborators selected events around Mrk 421 with $E > 100$ GeV. This avoids multiple scattering issues. They applied Johnson’s Inst.-2-Sky Transform.

Results:
1) Error Ellipses (sort of) point back towards Mrk 421
2) Smallest Ellipses consistent with limiting track resolution ($\approx \frac{\sigma_{\text{pitch}} \cdot \sqrt{2}}{60 \text{ cm}} \approx .01 \text{ deg.}$)
3) Covariant Localization ~2 better than PSF Localization
Fermi-LAT is a highly optimized HEP detector for detecting Gamma Rays in the Space Environment.

The Reconstruction of the Directions on the Sky were realized through 2nd order (covariance!).

A covariant analysis of source images will bring to bare the full power of the LAT.

Pass 8 will provide the “correct” covariance matrix using the Final Energy.

And… just perhaps we’ll finally see Pair Halos