Numerical integration of the chi-squared probability density function

Masaharu Hirayama

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1 Chi-squared probability function

This note illustrates a procedure to estimate the complement of the chi-squared probability function defined by

$$Q_n(x) = \int_x^{\infty} f_n(t)dt, \tag{1}$$

where n is the degrees of freedom of the chi-squared distribution of interest, x a value of χ^2 above which the chi-squared probability density function is to be integrated, and

$$f_n(t) = C_n e^{-\frac{t}{2}} t^{\frac{n}{2}-1} \tag{2}$$

$$C_n = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}. \tag{3}$$

The method of estimation illustrated here is to numerically integrate $f_n(t)$ over t as in Eq. 1. Also included are the estimates of the difference between the true value of the desired probability and the value estimated through the illustrated method.

2 Numerical integration

This section describes how to perform the integration of Eq. 1 numerically in detail. First, the estimators of $Q_n(x)$ are defined, such that the estimators give an upper limit and a lower limit of $Q_n(x)$. It gives a foundation of the numerical estimation of $Q_n(x)$ illustrated in this note. Second, the discrepancy between the estimators and $Q_n(x)$ is discussed. Finally, a procedure to estimate $Q_n(x)$ for a given precision is outlined.

2.1 Estimators of $Q_n(x)$

For $0 < x_1 < x_2$, define function

$$q_n(x_1, x_2) = \int_{x_1}^{x_2} f_n(t)dt. \tag{4}$$

For a given set of x_i $(i = 1, 2, \cdots, k)$ which satisfy $x_i < x_{i+1}$ for any i, if $x_1 = x$, it is trivial that

$$Q_n(x) = \sum_{i=1}^{k-1} q_n(x_i, x_{i+1}) + Q_n(x_k). \tag{5}$$

Now we define estimators of $Q_n(x)$ as

$$Q_n^+(x) = \sum_{i=1}^{k-1} q_n^+(x_i, x_{i+1}) + R_n(x_k) \tag{6}$$

$$Q_n^-(x) = \sum_{i=1}^{k-1} q_n^-(x_i, x_{i+1}), \tag{7}$$

where

$$q_n^+(x_1, x_2) = (x_2 - x_1) \cdot \max_{x_1 \le s \le x_2} f_n(s) \tag{8}$$

$$q_n^-(x_1, x_2) = (x_2 - x_1) \cdot \min_{x_1 \le s \le x_2} f_n(s) \tag{9}$$

$$R_n(x) = \frac{2x f_n(x)}{x - n}. \tag{10}$$

Since

$$f_n(t) \le \max_{x_1 \le s \le x_2} f_n(s) \tag{11}$$

for $x_1 \le t \le x_2$ by definition,

$$\begin{aligned} q_n(x_1, x_2) &\le \int_{x_1}^{x_2} \max_{x_1 \le s \le x_2} f_n(s)dt && (12) \\ &= \max_{x_1 \le s \le x_2} f_n(s) \int_{x_1}^{x_2} dt && (13) \\ &= (x_2 - x_1) \cdot \max_{x_1 \le s \le x_2} f_n(s) && (14) \\ &= q_n^+(x_1, x_2). && (15) \end{aligned}$$

Similarly,

$$q_n(x_1, x_2) \geq q_n^-(x_1, x_2). \tag{16}$$

From section 3.1, $Q_n(x_k) < R_n(x_k)$ for $x_k > n$. Combining that with Eqs. 5, 6, and 15, it follows that

$$Q_n(x) < Q_n^+(x). \tag{17}$$

for $x_k > n$. Since $f_n(t) > 0$ for $t > 0$, $Q_n(x) > 0$ for $x \geq 0$. Combining that with Eqs. 5, 7, and 16, it follows that

$$Q_n(x) > Q_n^-(x). \tag{18}$$

In summary, once our estimators, $Q_n^+(x)$ and $Q_n^-(x)$, are computed through their definitions, Eqs. 6, 7, 8, 9, and 10, they satisfy the following inequality as desired.

$$Q_n^-(x) < Q_n(x) < Q_n^+(x) \tag{19}$$

2.2 Computational procedure

Consider estimating $Q_n(x)$ for a given n and x by computing $Q_n^+(x)$ and $Q_n^-(x)$ as described in the previous section, to the precision of ϵ in fraction to the true value, $Q_n(x)$. In other words, our goal is to achieve

$$Q_n^+(x) - Q_n^-(x) < \epsilon \cdot Q_n(x) \tag{20}$$

by choosing an appropriate set of x_i $(i = 1, 2, \cdots, k)$.

From section 3.4, if x_i for $i = 1, 2, \cdots, k$ are defined by Eq. 78 and 79, then Eq. 77 holds. Therefore, letting $p = \epsilon/2$ gives

$$\frac{Q_n^+(x) - Q_n^-(x)}{Q_n(x)} \leq \frac{\epsilon}{2} + \frac{R_n(x_k)}{Q_n(x)}. \tag{21}$$

Also, if x_k is chosen such that $R_n(x_k) \leq (\epsilon/2) \cdot Q_n^-(x)$, then it implies that $R_n(x_k) < (\epsilon/2) \cdot Q_n(x)$, and hence $R_n(x_k)/Q_n(x) < \epsilon/2$. Therefore,

$$\frac{Q_n^+(x) - Q_n^-(x)}{Q_n(x)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \tag{22}$$

as we desired. Consequently a computational procedure to compute the estimators, $Q_n^+(x)$ and $Q_n^-(x)$, is as follows.

1. Set $x_1 = x$.

2. Set $Q_{max} = Q_{min} = 0.0$.

3. Repeat the following steps for $i = 1, 2, \cdots$ until $R_n(x_{i+1})$ becomes small enough to satisfy $R_n(x_{i+1}) \leq (\epsilon/2) \cdot Q_{min}$.

   (a) Compute x_{i+1} from x_i by Eq. 79 for $p = \epsilon/2$.

   (b) Compute $q_n^+(x_i, x_{i+1})$ and $q_n^-(x_i, x_{i+1})$ and add them to Q_{max} and Q_{min}, respectively.

$$Q_{max} = Q_{max} + q_n^+(x_i, x_{i+1}) \tag{23}$$
$$Q_{min} = Q_{min} + q_n^-(x_i, x_{i+1}) \tag{24}$$

   (c) Compute $R_n(x_{i+1})$.

   (d) If $x_{i+1} > n$ and $R_n(x_{i+1}) \leq (\epsilon/2) \cdot Q_{min}$, stop the iteration and set $k = i + 1$. Otherwise, repeat the above steps.

4. Add $R_n(x_k)$ to Q_{max}, namely, $Q_{max} = Q_{max} + R_n(x_k)$.

5. Now the estimators are successfully computed as $Q_n^+(x) = Q_{max}$ and $Q_n^-(x) = Q_{min}$, for which Eq. 20 holds.

3 Useful inequalities

This section lists the inequalities used in this note with their complete proofs.

3.1 Inequality A

For any $x > n$, the following inequality holds.

$$Q_n(x) < \frac{2x f_n(x)}{x - n} \tag{25}$$

Proof

For a given $x > n$, define function

$$g_n(t, x) = f_n(x) \left(\frac{x}{t}\right)^{\beta}, \tag{26}$$

where $\beta = \frac{1}{2}(x - n + 2)$. Since $x > n$, $x > 0$ and $\beta > 1$. Also define function

$$h_n(t,x) = \frac{f_n(t)}{g_n(t,x)} \tag{27}$$

$$= \exp\left\{-\frac{1}{2}(t-x)\right\}\left(\frac{t}{x}\right)^{\frac{n}{2}+\beta-1}. \tag{28}$$

It is trivial that $h_n(t,x) = 1$ at $t = x$, and that

$$\frac{d}{dt}h_n(t,x) = -\frac{1}{2x}\exp\left\{-\frac{1}{2}(t-x)\right\}\left(\frac{t}{x}\right)^{\frac{n}{2}+\beta-2}(t-x) \tag{29}$$

$$< 0 \tag{30}$$

for $t > x$. Therefore, $h_n(t,x) < 1$ for $t > x$. Since $g_n(t,x) > 0$ for $t > 0$, it implies that $f_n(t) < g_n(t,x)$ for $t > x$. Hence,

$$Q_n(x) < \int_x^\infty g_n(t,x)dt \tag{31}$$

$$= f_n(x)x^\beta \int_x^\infty t^{-\beta}dt \tag{32}$$

$$= f_n(x)x^\beta \cdot \frac{x^{1-\beta}}{\beta - 1} \tag{33}$$

$$= \frac{2xf_n(x)}{x-n}. \tag{34}$$

(End of proof)

3.2 Inequality B

For any $n > 2$ and any $x > n - 2$, the following inequality holds.

$$Q_n(x) < \frac{2xf_n(x)}{x-n+2} \tag{35}$$

Proof

For a given $n > 2$ and a given $x > n - 2$, define function

$$g_n(t,x) = f_n(x)e^{-\alpha(t-x)}, \tag{36}$$

where $\alpha = \frac{1}{2x}(x-n+2)$. Since $n > 2$ and $x > n-2$, $x > 0$ and $0 < \alpha < \frac{1}{2}$. Also define function

$$
\begin{aligned}
h_n(t,x) &= \frac{f_n(t)}{g_n(t,x)} &(37)\\
&= \exp\left\{(\alpha - \frac{1}{2})(t-x)\right\}\left(\frac{t}{x}\right)^{\frac{n}{2}-1}. &(38)
\end{aligned}
$$

It is trivial that $h_n(t,x) = 1$ at $t = x$, and that

$$
\begin{aligned}
\frac{d}{dt}h_n(t,x) &= \frac{2\alpha-1}{2x}\exp\left\{(\alpha-\frac{1}{2})(t-x)\right\}\left(\frac{t}{x}\right)^{\frac{n}{2}-2}(t-x) &(39)\\
&< 0 &(40)
\end{aligned}
$$

for $t > x$. Therefore, $h_n(t,x) < 1$ for $t > x$. Since $g_n(t,x) > 0$ for any t, it implies that $f_n(t) < g_n(t,x)$ for $t > x$. Hence,

$$
\begin{aligned}
Q_n(x) &< \int_x^\infty g_n(t,x)dt &(41)\\
&= f_n(x)\int_x^\infty e^{-\alpha(t-x)}dt &(42)\\
&= f_n(x)\cdot\frac{1}{\alpha} &(43)\\
&= \frac{2xf_n(x)}{x-n+2} &(44)
\end{aligned}
$$

(End of proof)

3.3 Inequality C

For any x_1 and any x_2 which satisfy $0 < x_1 < x_2$, if $x_1 \geq n-2$ or $x_2 \leq n-2$, the following inequality holds.

$$
\frac{q_n^+(x_1,x_2) - q_n^-(x_1,x_2)}{q_n(x_1,x_2)} \leq \frac{x_2-x_1}{2}\left|1-\frac{n-2}{\hat{x}}\right| \qquad (45)
$$

where

$$
\hat{x} = \begin{cases} x_2 & \text{for } n > 2 \text{ and } x_1 \geq n-2 \\ x_1 & \text{otherwise} \end{cases} \qquad (46)
$$

Proof

Define function

$$D_n(t) = (x_2 - x_1)\frac{1}{f_n(t)}\frac{d}{dt}f_n(t) \tag{47}$$

$$= (x_2 - x_1)\frac{d}{dt}\ln f_n(t) \tag{48}$$

$$= -\frac{x_2 - x_1}{2}\left(1 - \frac{n-2}{t}\right). \tag{49}$$

Consider t which satisfies $x_1 \leq t \leq x_2$. For $n > 2$, $D_n(t)$ monotonically decreases as t increases, hence

$$D_n(x_2) \leq D_n(t) \leq D_n(x_1). \tag{50}$$

For $x_1 \geq n-2$, $t \geq n-2$ also holds, hence $D_n(t) \leq 0$ when $n > 2$, thus

$$|D_n(t)| \leq |D_n(x_2)|. \tag{51}$$

For $x_2 \leq n-2$, $t \leq n-2$ also holds, hence $D_n(t) \geq 0$ when $n > 2$, thus

$$|D_n(t)| \leq |D_n(x_1)|. \tag{52}$$

Therefore, for $n > 2$,

$$|D_n(t)| \leq |D_n(\hat{x})|. \tag{53}$$

For $n = 1$, $D_1(t)$ monotonically increases as t, and is negative for any t, hence,

$$|D_1(t)| \leq |D_1(x_1)| = |D_1(\hat{x})|. \tag{54}$$

For $n = 2$, $D_2(t)$ is constant over t, hence

$$|D_2(t)| \leq |D_2(\hat{x})| \tag{55}$$

also holds. Thus, in summary,

$$|D_n(t)| \leq |D_n(\hat{x})| \tag{56}$$

holds for any n. Since $f_n(t) > 0$ for any t, it follows that

$$f_n(t)|D_n(t)| \leq f_n(t)|D_n(\hat{x})| \tag{57}$$

and hence,

$$\begin{aligned}\int_{x_1}^{x_2} f_n(t)|D_n(t)|dt &\leq \int_{x_1}^{x_2} f_n(t)|D_n(\hat{x})|dt && (58)\\ &= |D_n(\hat{x})|\int_{x_1}^{x_2} f_n(t)dt && (59)\\ &= |D_n(\hat{x})|q_n(x_1,x_2). && (60)\end{aligned}$$

Since $x_1 < x_2$, it is trivial that

$$|D_n(t)| = \begin{cases} D_n(t) & \text{for } n > 2 \text{ and } t \leq n-2 \\ -D_n(t) & \text{otherwise.} \end{cases} \quad (61)$$

Therefore, for $n > 2$ and $x_2 \leq n-2$,

$$\begin{aligned}\int_{x_1}^{x_2} f_n(t)|D_n(t)|dt &= \int_{x_1}^{x_2} f_n(t)D_n(t)dt && (62)\\ &= (x_2-x_1)\int_{x_1}^{x_2} \frac{d}{dt}f_n(t)dt && (63)\\ &= (x_2-x_1)\{f_n(x_2)-f_n(x_1)\}. && (64)\end{aligned}$$

Since $f_n(t)$ monotonically increases as t for $n > 2$ and $t \leq n-2$,

$$\begin{aligned}\max_{x_1\leq s\leq x_2} f_n(s) &= f_n(x_2) && (65)\\ \min_{x_1\leq s\leq x_2} f_n(s) &= f_n(x_1). && (66)\end{aligned}$$

Therefore,

$$(x_2-x_1)\{f_n(x_2)-f_n(x_1)\} = q_n^+(x_1,x_2) - q_n^-(x_1,x_2). \quad (67)$$

and hence,

$$\int_{x_1}^{x_2} f_n(t)|D_n(t)|dt = q_n^+(x_1,x_2) - q_n^-(x_1,x_2). \quad (68)$$

Similarly, for $n \leq 2$ or $x_1 \geq n-2$,

$$\begin{aligned}\int_{x_1}^{x_2} f_n(t)|D_n(t)|dt &= \int_{x_1}^{x_2} f_n(t)\{-D_n(t)\}dt && (69)\\ &= -(x_2-x_1)\int_{x_1}^{x_2} \frac{d}{dt}f_n(t)dt && (70)\\ &= (x_2-x_1)\{f_n(x_1)-f_n(x_2)\}, && (71)\end{aligned}$$

and $f_n(t)$ monotonically decreases as t increases, thus

$$\max_{x_1 \leq s \leq x_2} f_n(s) = f_n(x_1) \tag{72}$$

$$\min_{x_1 \leq s \leq x_2} f_n(s) = f_n(x_2). \tag{73}$$

Therefore,

$$(x_2 - x_1)\{f_n(x_1) - f_n(x_2)\} = q_n^+(x_1, x_2) - q_n^-(x_1, x_2), \tag{74}$$

and hence Eq. 68 also holds for the this case ($n \leq 2$ or $t \geq n-2$), and thus it holds for any n if $x_1 \geq n-2$ or $x_2 \leq n-2$.
Combining Eqs. 60 and 68 gives

$$q_n^+(x_1, x_2) - q_n^-(x_1, x_2) \leq |D_n(\hat{x})| q_n(x_1, x_2), \tag{75}$$

which holds for any n if $x_1 \geq n-2$ or $x_2 \leq n-2$. Since $q_n(x_1, x_2) > 0$ for $0 < x_1 < x_2$, it follows that

$$\frac{q_n^+(x_1, x_2) - q_n^-(x_1, x_2)}{q_n(x_1, x_2)} \leq |D_n(\hat{x})|. \tag{76}$$

(End of proof)

3.4 Inequality D

For any $p > 0$ and any $x > 0$, inequality

$$\frac{Q_n^+(x) - Q_n^-(x)}{Q_n(x)} \leq p + \frac{R_n(x_k)}{Q_n(x)} \tag{77}$$

holds for a series of x_i $(i = 1, 2, \cdots, k)$ defined by

$$x_1 = x \tag{78}$$

and

$$x_{i+1} = \begin{cases} x_i + p \cdot \eta_n(x_i) & \text{for} \quad n = 1 \\ x_i + 2p & \text{for} \quad n = 2 \\ x_i - p \cdot \eta_n(x_i) & \text{for} \quad n > 2 \text{ and } x_i < n-2 \\ & \qquad \text{and } x_i - p \cdot \eta_n(x_i) < n-2 \\ n-2 & \text{for} \quad n > 2 \text{ and } x_i < n-2 \\ & \qquad \text{and } x_i - p \cdot \eta_n(x_i) \geq n-2 \\ x_i + p \cdot \eta_n(\tilde{x}_i) & \text{for} \quad n > 2 \text{ and } x_i \geq n-2 \end{cases} \tag{79}$$

where

$$\begin{aligned}\eta_n(s) &= \frac{2s}{s-n+2}, &(80)\\ \tilde{x}_i &= x_i + 2p + \sqrt{2p(n-2)}. &(81)\end{aligned}$$

Proof

From Eqs. 6 and 7, it is trivial that

$$Q_n^+(x) - Q_n^-(x) - R_n(x_k) = \sum_{i=1}^{k-1}\{q_n^+(x_i, x_{i+1}) - q_n^-(x_i, x_{i+1})\}. \quad (82)$$

Since $x_i > 0$, it follows that $x_i > n-2$ for $n=1$ and for $n=2$. Therefore Eq. 45 holds for the cases, namely,

$$q_n^+(x_i, x_{i+1}) - q_n^-(x_i, x_{i+1}) \le \frac{x_{i+1}-x_i}{2}\left|1 - \frac{n-2}{\hat{x}_i}\right| q_n(x_i, x_{i+1}) \quad (83)$$

where

$$\hat{x}_i = \begin{cases} x_{i+1} & \text{for } n > 2 \text{ and } x_i \ge n-2 \\ x_i & \text{otherwise,} \end{cases} \quad (84)$$

since $q_n(x_i, x_{i+1}) > 0$ for $x_i > 0$ and $x_{i+1} > 0$, which is trivial from $x > 0$ and Eqs. 78 and 79.
Eq. 83 also holds for $n > 2$ and $x_i \ge n-2$ since the case satisfies one of the conditions for Eq. 45, or $x_1 \ge n-2$ in the notation for Eq. 45.
For $n > 2$ and $x_i < n-2$, if $x_i - p \cdot \eta_n(x_i) \ge n-2$, then $x_{i+1} = n-2$, thus x_{i+1} satisfies the other condition for Eq. 45, or $x_2 \le n-2$ in the notation for Eq. 45. Therefore Eq. 83 also holds for the case. If $x_i - p \cdot \eta_n(x_i) < n-2$ for $n > 2$ and $x_i < n-2$, then

$$\begin{aligned} x_{i+1} &= x_i - p \cdot \eta_n(x_i) &(85)\\ &= x_i - \frac{2px_i}{x_i - n + 2} &(86)\\ &< x_i - (x_i - n + 2) &(87)\\ &= n-2 &(88)\end{aligned}$$

noting that $x_i - n + 2 < 0$. Therefore x_{i+1} satisfies the latter condition for Eq. 45, or $x_2 \le n-2$ in the notation for Eq. 45, and Eq. 83 holds for the case, too. In conclusion, Eq. 83 holds for all the cases which appear in Eq. 79. Combining Eqs. 82 and 83 implies

$$Q_n^+(x) - Q_n^-(x) - R_n(x_k) \leq \sum_{i=1}^{k-1} \frac{x_{i+1} - x_i}{2} \left| 1 - \frac{n-2}{\hat{x}_i} \right| q_n(x_i, x_{i+1}), \tag{89}$$

which holds for a series of x_i defined by Eq. 79.
For $n = 1$, $x_{i+1} - x_i = p \cdot \eta_n(x_i)$ and $\hat{x}_i = x_i$, therefore,

$$\begin{aligned}
\frac{x_{i+1} - x_i}{2} \left| 1 - \frac{n-2}{\hat{x}_i} \right| &= \frac{p \cdot \eta_n(x_i)}{2} \left| 1 + \frac{1}{x_i} \right| && (90)\\
&= p \cdot \frac{x_i}{x_i + 1} . \left(1 + \frac{1}{x_i} \right) && (91)\\
&= p && (92)
\end{aligned}$$

For $n = 2$, $x_{i+1} - x_i = 2p$ and $\hat{x}_i = x_i$, therefore,

$$\frac{x_{i+1} - x_i}{2} \left| 1 - \frac{n-2}{\hat{x}_i} \right| = p \tag{93}$$

For $n > 2$ and $x_i < n - 2$ and $x_i - p \cdot \eta_n(x_i) < n - 2$, $x_{i+1} - x_i = -p \cdot \eta_n(x_i)$ and $\hat{x}_i = x_i$, therefore,

$$\begin{aligned}
\frac{x_{i+1} - x_i}{2} \left| 1 - \frac{n-2}{\hat{x}_i} \right| &= -\frac{p \cdot \eta_n(x_i)}{2} \left| 1 - \frac{n-2}{x_i} \right| && (94)\\
&= -\frac{p x_i}{x_i - n + 2} \left(\frac{n-2}{x_i} - 1 \right) && (95)\\
&= p && (96)
\end{aligned}$$

For $n > 2$ and $x_i < n - 2$ and $x_i - p \cdot \eta_n(x_i) \geq n - 2$, $x_{i+1} - x_i = n - 2 - x_i$ and $\hat{x}_i = x_i$, therefore,

$$\begin{aligned}
\frac{x_{i+1} - x_i}{2} \left| 1 - \frac{n-2}{\hat{x}_i} \right| &= \frac{n - 2 - x_i}{2} \left| 1 - \frac{n-2}{x_i} \right| && (97)\\
&\leq -\frac{p \cdot \eta_n(x_i)}{2} \left(\frac{n-2}{x_i} - 1 \right) && (98)\\
&= -\frac{p x_i}{x_i - n + 2} \left(\frac{n-2}{x_i} - 1 \right) && (99)\\
&= p && (100)
\end{aligned}$$

For $n > 2$ and $x_i \geq n-2$, $x_{i+1} - x_i = p \cdot \eta_n(\tilde{x}_i)$ and $\hat{x}_i = x_{i+1}$, therefore,

$$\begin{aligned}
\frac{x_{i+1}-x_i}{2}\left|1-\frac{n-2}{\hat{x}_i}\right| &= \frac{p\cdot\eta_n(\tilde{x}_i)}{2}\left|1-\frac{n-2}{x_{i+1}}\right| && (101)\\
&= p\cdot\left(1-\frac{n-2}{\tilde{x}_i}\right)^{-1}\left(1-\frac{n-2}{x_{i+1}}\right) && (102)\\
&= p\cdot\left(1-\frac{n-2}{x_{i+1}}\cdot\frac{\tilde{x}_i-x_{i+1}}{\tilde{x}_i-n+2}\right) && (103)\\
&< p && (104)
\end{aligned}$$

since $\tilde{x}_i > x_{i+1}$, because

$$\begin{aligned}
\tilde{x}_i - x_{i+1} &= 2p+\sqrt{2p(n-2)} - p\eta_n(x_i+2p+\sqrt{2p(n-2)}) && (105)\\
&= \frac{2p+\sqrt{2p(n-2)}}{x_i+2p+\sqrt{2p(n-2)}-n+2}\cdot(2p+x_i-n+2) && (106)\\
&> 0. && (107)
\end{aligned}$$

Combining Eqs. 89, 92, 93, 96, 100, and 104 give

$$\begin{aligned}
Q_n^+(x) - Q_n^-(x) - R_n(x_k) &\leq \sum_{i=1}^{k-1} p\cdot q_n(x_i,x_{i+1}) && (108)\\
&= p\sum_{i=1}^{k-1} q_n(x_i,x_{i+1}) && (109)\\
&= p\left\{Q_n(x)-R_n(x_k)\right\} && (110)\\
&\leq pQ_n(x) && (111)
\end{aligned}$$

Therefore, noting that $Q_n(x) > 0$ for $x > 0$,

$$\frac{Q_n^+(x)-Q_n^-(x)-R_n(x_k)}{Q_n(x)} \leq p. \qquad (112)$$

(End of proof)

Supplement: choice of $\tilde{x}_i$

As trivial from the above proof, the question here is to find x_{i+1} which satisfies $x_{i+1} > x_i$ and

$$\frac{x_{i+1}-x_i}{2}\left(1-\frac{n-2}{x_{i+1}}\right) \leq p \qquad (113)$$

for a given x_i when $x_i \geq n-2$ and $n > 2$. Eq. 113 can be written as

$$x_{i+1} - x_i \leq p \cdot \eta_n(x_{i+1}), \tag{114}$$

noting that $\eta_n(x_{i+1}) > 0$ since $x_{i+1} > x_i \geq n-2$. Consider $\tilde{x_i}$ which satisfies

$$\tilde{x_i} \geq x_i + p \cdot \eta_n(\tilde{x_i}). \tag{115}$$

If $\tilde{x_i}$ can be found, Eq. 114 holds for x_{i+1} defined as

$$x_{i+1} = x_i + p \cdot \eta_n(\tilde{x_i}), \tag{116}$$

because, noting that $\eta_n(s)$ decreases as s increases,

$$\begin{aligned} x_{i+1} - x_i &= p \cdot \eta_n(\tilde{x_i}) && (117)\\ &\leq p \cdot \eta_n(x_i + p \cdot \eta_n(\tilde{x_i})) && (118)\\ &= p \cdot \eta_n(x_{i+1}). && (119) \end{aligned}$$

Therefore, the question here is to find $\tilde{x_i}$ which satisfies Eq. 115. By defining Δ by

$$\Delta = \tilde{x_i} - x_i, \tag{120}$$

Eq. 115 becomes

$$\Delta \geq p \cdot \eta_n(x_i + \Delta), \tag{121}$$

or

$$\Delta^2 + (x_i - n + 2 - 2p)\Delta - 2px_i \geq 0. \tag{122}$$

The solution for $\Delta > 0$ is

$$\Delta \geq \Delta_{min} \tag{123}$$

where

$$\begin{aligned} \Delta_{min} &= -\frac{1}{2}(x_i - n + 2 - 2p) \\ &\quad + \frac{1}{2}\sqrt{(x_i - n + 2 + 2p)^2 + 8p(n-2)} && (124)\\ &< 2p + \sqrt{2p(n-2)} && (125) \end{aligned}$$

since

$$\begin{aligned} &\sqrt{(x_i - n + 2 + 2p)^2 + 8p(n-2)} \\ &< (x_i - n + 2 + 2p) + \sqrt{8p(n-2)} && (126) \end{aligned}$$

for $p > 0$ and $n > 2$ and $x_i \geq n-2$. Therefore, by letting $\Delta = 2p + \sqrt{2p(n-2)}$, Eq. 121 holds since $\Delta > \Delta_{min}$, and hence Eq. 113 holds for x_{i+1} defined by Eqs. 116 and 120.

4 Complete gamma function

Computation of the complete gamma function that appears in Eq. 3 is straightforward. For $n = 1$,

$$\begin{aligned} \Gamma\left(\tfrac{n}{2}\right) &= \Gamma\left(\tfrac{1}{2}\right) \\ &= \sqrt{\pi}, \end{aligned} \tag{127}$$

and for $n = 2$,

$$\begin{aligned} \Gamma\left(\tfrac{n}{2}\right) &= \Gamma(1) \\ &= 1. \end{aligned} \tag{128}$$

Letting m a positive integer, or $m = 1, 2, \cdots$, for an odd integer $n = 2m + 1$,

$$\begin{aligned} \Gamma\left(\tfrac{n}{2}\right) &= \Gamma\left(m + \tfrac{1}{2}\right) \\ &= \left(m - \tfrac{1}{2}\right)\left(m - \tfrac{3}{2}\right)\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\Gamma\left(\tfrac{1}{2}\right) \\ &= \left(m - \tfrac{1}{2}\right)\left(m - \tfrac{3}{2}\right)\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\sqrt{\pi}, \end{aligned} \tag{129}$$

and for an even integer $n = 2m + 2$,

$$\begin{aligned} \Gamma\left(\tfrac{n}{2}\right) &= \Gamma(m+1) \\ &= m(m-1)\cdots 2\cdot 1\cdot\Gamma(1) \\ &= m(m-1)\cdots 2\cdot 1. \end{aligned} \tag{130}$$

In summary, for $m = 1, 2, \cdots$,

$$\Gamma\left(\frac{n}{2}\right) = \prod_{k=1}^{m-1}\left(\frac{n}{2} - k\right) \times \begin{cases} 1 & \text{for} \quad n = 2m \\ \sqrt{\pi} & \text{for} \quad n = 2m - 1 \end{cases} \tag{131}$$

or in a logarithmic form

$$\ln\Gamma\left(\frac{n}{2}\right) = \sum_{k=1}^{m-1}\ln\left(\frac{n}{2} - k\right) + \begin{cases} 0 & \text{for} \quad n = 2m \\ \frac{1}{2}\ln\pi & \text{for} \quad n = 2m - 1. \end{cases} \tag{132}$$